

# TORSION HOMOLOGY GROWTH BEYOND ASYMPTOTICS

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**ABSTRACT.** We show that (under mild assumptions) the generating function of log homology torsion of a knot exterior has a meromorphic continuation to the entire complex plane. As corollaries, this gives new proofs of (a) the Silver–Williams asymptotic, (b) Fried’s theorem on reconstructing the Alexander polynomial (c) Gordon’s theorem on periodic homology. Our results generalize to other rank 1 growth phenomena, e.g. Reidemeister–Franz torsion growth for higher-dimensional knots. We also analyze the exceptional cases where the meromorphic continuation does not exist.

Let  $K \subset S^3$  be a knot and  $X_K := S^3 - K$  its knot complement. Write  $X_r$  for the  $r$ -th cyclic covering of  $X_K$ . The Silver–Williams theorem asserts that

$$(0.1) \quad \lim_{r \rightarrow \infty} \frac{\log |H_1(X_r, \mathbf{Z})_{\text{tor}}|}{r} = \log \mathcal{M}(\Delta_K),$$

if  $\mathcal{M}(\Delta_K) > 1$  is the Mahler measure of the Alexander polynomial  $\Delta_K$  of the knot. Instead of just asking about the asymptotic behaviour of torsion homology growth in  $H_1$ , we could ask about *all* values  $H_1(X_r, \mathbf{Z})_{\text{tor}}$ . Define

$$(0.2) \quad E(z) := \sum_{r \geq 1} \log |H_1(X_r, \mathbf{Z})_{\text{tor}}| \cdot z^r.$$

This is a power series around  $z = 0$ . A heuristic argument shows that the Silver–Williams asymptotics suggest that  $E$  might have a meromorphic continuation beyond the unit circle with a pole of order 1 or 2 at  $z = 1$ . Indeed, whenever  $E$  has said property, the asymptotics of Equation 0.1 are an immediate consequence. Inspired by this, we seek to understand whether  $E$  has such a meromorphic continuation.

Let us call a root  $\beta$  of  $\Delta_K$  *diophantine* if it lies on the unit circle, but is not a root of unity.

**Theorem.** *Suppose the Alexander polynomial  $\Delta_K$  of a knot has no diophantine roots. Then  $E$  admits a meromorphic continuation to the entire complex plane.*

(1) *The pole locus is*

$$\{\beta^n \mid \Delta_K(\beta) = 0 \text{ and } n \text{ an integer}\} \setminus (\text{open unit disc}).$$

*For each pole, its residue encodes the multiplicity of  $\beta$  as a root of  $\Delta_K$ .*

(2) *At  $z = 1$ , it has a pole of order 1 or 2. All other poles have order 1.*

This is our first main result (Theorem 8.2). If the (rather mild) assumptions of the theorem are met, it affirms our heuristic about a pole at  $z = 1$ , and in fact it gives a new proof of the Silver–Williams asymptotic. But it implies more. A theorem of Fried says that the knowledge of the torsion orders  $|H_1(X_r, \mathbf{Z})_{\text{tor}}|$  for all  $r$  allows us to reconstruct

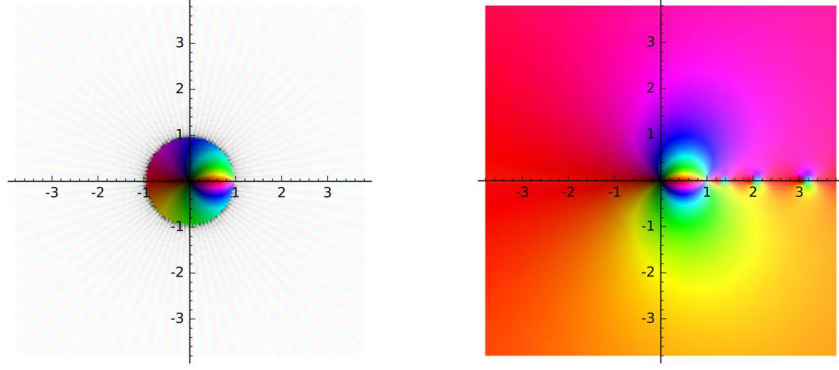
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*Date:* February 22, 2017.

The author was supported by the DFG GK1821 “Cohomological Methods in Geometry”.

the Alexander polynomial of the knot. However, this also follows at once from the above theorem because we just need to look at the poles of the meromorphic continuation.

The theorem is not just a theoretical result. We can explicitly compute this analytic continuation. For example, for the knot “K8<sub>256</sub>”<sup>1</sup> we get



On the left, we see the evaluation of  $E$  as the power series of line 0.2. One can clearly see how the series  $E$  diverges outside the unit circle (as is forced by the pole at  $z = 1$ ). On the right, we see our analytic continuation. The Alexander polynomial has roots at  $\frac{2}{3}$  and  $\frac{3}{2}$ , and its smallest integer powers outside the unit disc are at 1.5, 2.25, 3.37..., as one can also read off the plot.

We also obtain a strengthening of Gordon’s classical result on periodic torsion homology [Gor72]:

**Theorem.** *For a given knot, the following are equivalent:*

- (1) *The values  $|H_1(X_r, \mathbf{Z})_{\text{tor}}|$  are periodic in  $r$ .*
- (2) *All Alexander roots are roots of unity.*
- (3)  *$E$  is a rational function.*
- (4)  *$E$  has an analytic continuation to the entire complex plane with only finitely many poles.*
- (5) *The values  $\log |H_1(X_r, \mathbf{Z})_{\text{tor}}|$  satisfy a linear recurrence equation.*

This will be Theorem 8.11. The equivalence (1)  $\Leftrightarrow$  (2) is Gordon’s classical result.

So far, we have described results under the assumption that no root of the Alexander polynomial is diophantine. What happens in the rare case if there is a diophantine root? In this case, everything changes drastically. We prove:

**Theorem.** *Suppose the Alexander polynomial  $\Delta_K$  of a knot has at least one diophantine root. Then  $E$  has the unit circle as its natural boundary, i.e. no analytic continuation is possible. Moreover,*

$$\lim_{z \rightarrow p} (1 - |z|)E(z) \neq 0$$

*if  $|p| = 1$  lies in the multiplicative span of the diophantine roots, and it is zero if  $p$  is multiplicatively independent of all diophantine roots.*

Here “ $\lim_{z \rightarrow p}$ ” refers to the limit under all sequences of constant complex argument. This will be Theorem 8.7.

In fact, we get a relation between the singular values on the unit circle and special  $L$ -values, see §0.3.

<sup>1</sup>census tabulation along size of triangulation as used in SnapPea, [CKM14]

**0.1. Application to Reidemeister torsion.** As mentioned before, our results do not just apply to knots in the 3-sphere, but also to other rank one growth phenomena governed by the Mahler measure. As is explained in many places, e.g. [Tur86], it is natural to view the torsion homology in  $H_1$  in the Silver–Williams theorem as a special instance of the growth of Reidemeister–Franz torsion.

For example, using a variation of the arguments of our Theorems, we can also show the following:

**Theorem.** *Let  $K^n \subset M^{n+2}$  be an  $n$ -knot, where  $M^{n+2}$  is a  $(n+2)$ -dimensional homology sphere (in the PL category). If  $\Delta_{K^n,i}$  denotes the  $i$ -th Alexander polynomial, and none of the  $\Delta_{K^n,i}$  has a root in  $\mu_\infty$ , then the generating function of the Reidemeister torsion*

$$J_{K^n}(z) := \sum_{r \geq 1} \log(\tau_r) \cdot z^r$$

with

$$\tau_r := \prod_{i=1}^n \left| H_i(\widehat{X}_r, \mathbf{Z}) \right|^{(-1)^{i+1}},$$

where  $\widehat{X}_r$  is the  $r$ -th cyclic branched covering, has the following property:

- (1) *If no root of any of the Alexander polynomials  $\Delta_{K^n,i}$  has absolute value 1, the function admits a meromorphic continuation to the entire complex plane. Its poles are located at most at all integer powers of all roots of all  $\Delta_{K^n,i}$  which lie outside the open unit disc.*
- (2) *If some  $\Delta_{K^n,i}$  has a root of absolute value 1 and no other  $\Delta_{K^n,j}$  (with  $j \neq i$ ) has a root at the same value, then  $J_{K^n}$  has the unit circle as its natural boundary. An analytic continuation beyond the unit circle is impossible.*

See Theorem 8.12. This result arises immediately from combining Porti’s Mayberry–Murasugi type formula of [Por04] with the tools which we develop in this paper.

Many similar variations around Reidemeister torsion will be possible.

There are also applications which are less connected to geometry. For example, Hillar [Hil05] studied polynomials which have the same cyclic resultants:

**Theorem** (Hillar). *Let  $f, g \in \mathbf{R}[t]$  be polynomials such that their cyclic resultants are all non-zero. Then the absolute values of the cyclic resultants agree if and only if there exist  $u, v \in \mathbf{C}[t]$  with  $u(0) \neq 0$  and integers  $\ell_1, \ell_2 \geq 0$  such that*

$$\begin{aligned} f(t) &= \pm t^{\ell_1} v(t) u(t^{-1}) t^{\deg u} \\ g(t) &= t^{\ell_2} v(t) u(t). \end{aligned}$$

For polynomials which have no roots on the unit circle (this is the generic case), our methods give a new proof of this theorem. See §8.3.

**0.2. Open questions.** We do not know how the generation functions behave in rank  $d \geq 2$  situations,

$$\pi_1(X, *) \rightarrow \mathbf{Z}^d,$$

as in Le [Le14] and Raimbault [Rai12], where one also has an asymptotic governed by the Mahler measure. More broadly, one could dream about studying the generating functions of torsion coming from lattice quotients in Lie groups, inspired by the asymptotics à la [BV13]. Unfortunately, at present, this seems completely out of reach.

**0.3. Relation to special  $L$ -values.** As an accidental finding along the way, we find a new relation to special  $L$ -values. So far, it is known that there is *some* relation between Mahler measures and special  $L$ -values through the Beilinson conjectures. A popular example is the two-variable Mahler measure

$$\mathcal{M}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi),$$

where  $\chi$  is a certain Dirichlet character. This was discovered by Smyth, and later theoretically explained by Deninger [Den97]. However, to the best of my knowledge, this was the only suggestion of a possible connection between the Silver–Williams theorem and special  $L$ -values.

However, when evaluating the singular limit values of  $E$  for a knot with diophantine roots, other special  $L$ -values at  $s = 1$  show up:

**Theorem.** *Let  $K \subset S^3$  be a knot and  $\Delta_K$  its Alexander polynomial. Suppose  $\Delta_K$  has at least one diophantine root. Let  $p$  be a point of the unit circle which lies in the multiplicative span<sup>2</sup> of the diophantine roots of  $\Delta_K$ . Then*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)E(z) \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

where  $\chi$  runs through a finite set (depending on  $p$ ) of non-principal Dirichlet characters of various moduli.

In §6 we provide (complicated) formulae which allow us the explicit evaluation of these limits. I have no philosophical explanation why special  $L$ -values show up in this context. It is mysterious. See Theorem 8.7.

**0.4. Technical results of independent interest.** In order to prove our main theorems, we need to establish various results which might be interesting in their own right – and a priori have little to do with torsion homology growth.

The principal result in this direction is an evaluation of certain time averages of ergodic nature:

**Theorem.** *Suppose  $\theta$  is a real number such that either*

- $e^{2\pi i \theta}$  is an algebraic number, or
- $\theta$  is badly approximable.

*Then the following holds:*

- (1) *If  $\dim_{\mathbf{Q}} \langle 1, \theta \rangle = 2$ : For all  $m \in \mathbf{Z}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| \cdot e^{2\pi i m n \theta} = -\frac{1}{2|m|} \delta_{m \neq 0}.$$

*If  $m \in \mathbf{Q} \setminus \mathbf{Z}$ , we get a value*

$$C_m \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

*where  $\chi$  ranges over a set of non-principal Dirichlet characters modulo  $2v$  for  $v \geq 1$  the denominator of  $m$  in lowest terms. The values  $C_m$  only depend on  $m$ , and are independent of  $\theta$ .*

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<sup>2</sup>We understand multiplicative dependency as allowing rational exponents, e.g. it is fine to take some  $l$ -th root of one of the diophantine roots. See Definition 8.6.

(2) If  $\alpha$  is a real number and  $\dim_{\mathbf{Q}} \langle 1, \theta, \alpha \rangle = 3$ , then for all  $m \in \mathbf{Z}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| \cdot e^{2\pi i m n \alpha} = 0.$$

This will be Theorem 6.1. We will use this theorem in order to understand the behaviour of the function  $E$  for knots whose Alexander polynomial has a diophantine root.

The proof is based on a (very strong) version of Weyl Equidistribution due to Baxa and Schoißengeier [BS02]. Their result is only available in dimension one, but we also need a two-variable version. For our purposes, a rather minimalistic extension of their proof is sufficient. It is just about strong enough to treat the computation which we need. This formulation might be of independent interest:

**Theorem.** (Baxa–Schoißengeier-type Equidistribution) *Suppose  $1, \theta_1, \dots, \theta_d$  are  $\mathbf{Q}$ -linearly independent real numbers. Suppose  $F \subseteq [0, 1] \cap \mathbf{Q}$  is finite. Suppose  $f : [0, 1]^d \rightarrow \mathbf{R}$  is a function in Class  $\text{BSU}^d(F)$  which admits a singular weight  $g$  (see Definition 5.11 in the main body of the text) such that*

$$\lim_{n \rightarrow \infty} \frac{g(\{n\theta_1\})}{n} = 0.$$

Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta_1\}, \dots, \{n\theta_d\}) = \int_{[0,1]^d} f(\underline{s}) \, d\underline{s}.$$

See Theorem 5.12. The proof is a mild variation of the method of [BS02]. For  $d = 1$ , we get nothing new.

**Acknowledgement.** *I heartily thank G. Wüstholtz for his suggestions on how to use equidistribution techniques, based on an earlier version of this manuscript in the spring of 2016. I thank William Stein and the SAGE team.*

## 1. HEURISTICS AND MOTIVATION

As we had explained in the introduction, the Silver–Williams asymptotic (which in the case of knots was developed in earlier work of González-Acuña and Short [GAnS91] as well as Riley [Ril90])

$$\lim_{r \rightarrow \infty} \frac{\log |H_1(X_r, \mathbf{Z})_{\text{tor}}|}{r} = \log \mathcal{M}(\Delta_K)$$

suggests that the function

$$E(z) = \sum_{r \geq 1} \log |H_1(X_r, \mathbf{Z})_{\text{tor}}| \cdot z^r$$

might have a meromorphic continuation to some disc of radius  $> 1$  with, likely, a pole of order 2 at  $z = 1$ . As we shall prove something stronger later, let us only sketch (under simplifying assumptions) why this speculation is natural. In rigorous form, we observe the following:

**Lemma 1.1.** *Suppose a power series*

$$f(z) := \sum_{r \geq 1} \log(a_r) \cdot z^r$$

converges and admits a meromorphic continuation to some disc of radius  $> 1$  with a single pole at  $z = 1$  of order  $N + 1$  and Laurent expansion

$$f(z - 1) = \frac{\log C}{(z - 1)^{N+1}} + \text{higher order terms}$$

at  $z = 1$ . If we additionally know that the limit

$$L := \lim_{r \rightarrow \infty} \frac{\log(a_r)}{r} \quad (\text{a “Silver–Williams asymptotic”})$$

exists, we must have  $N = 0$  or  $1$ . If  $N = 0$ , the limit is  $L = 0$ , and if  $N = 1$ , it is  $\log C$ .

*Proof.* Suppose  $R > 1$  is within the disc of meromorphic continuation. For all  $r \geq 0$ , the Residue Theorem implies that

$$\begin{aligned} \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta^{r+1}} d\zeta &= \log(a_r) + \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\log C}{(\zeta - 1)^{N+1} \zeta^{r+1}} d\zeta \\ &= \log(a_r) + (-1)^N \log(C) \cdot r(r-1) \cdots (r-N+1) \\ &\quad + (\text{degree} < N \text{ polynomial in } r). \end{aligned}$$

Thus,  $\log(a_r)$  is a degree  $\leq N$  polynomial in the variable  $r$  up to an error term which can be bounded by

$$\left| \int_{|\zeta|=R} \frac{E(\zeta)}{\zeta^{r+1}} d\zeta \right| \leq 2\pi R \cdot \int \frac{|E(\zeta)|}{R^{r+1}} d\zeta = \frac{2\pi}{R^r} \cdot \text{const.}$$

The constant depends on  $R$ , but not on  $r$ . Thus, since  $R > 1$ , so that  $\frac{2\pi}{R^r}$  converges to zero as  $r \rightarrow +\infty$ , we obtain

$$\lim_{r \rightarrow \infty} \frac{\log(a_r)}{r} = -(-1)^N \log(C) \lim_{r \rightarrow \infty} \frac{r(r-1) \cdots (r-N+1)}{r}.$$

Hence, if the limit on the left-hand side exists at all, we must have  $N = 0$  or  $1$ . If  $N = 0$ , the limit is zero, and if  $N = 1$ , it is  $\log C$ .  $\square$

In summary: The Silver–Williams asymptotic hints at the fact that *some* analytic continuation might exist. Theorem 8.2 will then settle this (for a generic knot).

## 2. PREPARATIONS

Let  $\mu_r \subset \mathbf{C}$  denote the set of all  $r$ -th roots of unity,  $\mu_\infty$  all roots of unity.

**Definition 2.1.** Let  $x \in \mathbf{C}$  be given. Define, at first formally, a complex power series

$$R_x(z) := \sum'_{r \geq 1} \log|1 - x^r| \cdot z^r,$$

where the notation  $\sum'$  means: We omit the  $r$ -th summand if  $x^r = 1$ .

Before we can continue, we need the following deep result of Gelfond:

**Proposition 2.2** (Gelfond). Let  $\alpha$  be an algebraic number with  $|\alpha| = 1$  and which is not a root of unity. Then there exist real numbers  $A, B > 0$  such that  $|\alpha^n - 1| > An^{-B}$  holds for all  $n \geq 1$ .

This version is sufficiently strong for our purposes. Nevertheless, much stronger results are known. See Baker–Wüstholz [BW93] for a concrete estimate (also involving the Mahler measure), or one of the several papers that have appeared since and improve on these bounds (e.g. [Lau08]).

**Lemma 2.3.** *Suppose  $x \in \mathbf{C}$  is a complex number.*

- (1) *If  $|x| < 1$ , then  $|\log |1 - x^r|| \leq 2|x|^r$  for all sufficiently large real numbers  $r$ .*
- (2) *If  $|x| > 1$ , then  $|\log |1 - x^r|| < 1 + r \log |x|$  for all sufficiently large real numbers  $r$ .*
- (3) *If  $|x| = 1$  and  $x$  is an algebraic integer, but not a root of unity, then there exists a constant  $C_x > 0$  such that  $|\log |1 - x^r|| \leq C_x \cdot \log(r)$  for all sufficiently large natural numbers  $r$ .*

The first two claims of the lemma are a harmless exercise, the third part depends on Gelfond's result.

**Lemma 2.4.** *Let  $x \in \mathbf{C}^\times$  be a given algebraic number and suppose  $x$  is not a root of unity. Then the series  $R_x$  has radius of convergence  $\geq 1$ . If  $|x| < 1$ , it even has radius of convergence  $\geq |x|^{-1} > 1$ .*

*Proof.* This follows directly from Lemma 2.3 and  $\lim_{r \rightarrow \infty} \sqrt[r]{\log(r)} = 1$ . □

The case where  $x$  is a root of unity is the only case where a meromorphic continuation of  $R_x$  to the entire complex plane is easy to achieve, so let us handle this case right away.

Below, we will write  $(\dots)$  to denote a holomorphic term.

**Proposition 2.5.** *Let  $x \in \mu_\infty$  be some root of unity. Then  $R_x$  admits a meromorphic continuation to the entire complex plane, given by*

$$R_x(z) = \sum_{l=1}^{m-1} \log |1 - \zeta_m^l| \frac{z^l}{1 - z^m},$$

where  $m$  denotes the (primitive) order of  $x$  and  $\zeta_m := x$ . Near  $z = 1$ ,  $R_x$  has the Laurent expansion

$$\begin{aligned} &= \frac{1}{m} \log \left( \frac{1}{m} \right) \frac{1}{z-1} \\ &- \left( \frac{m-1}{2} \frac{1}{m} \log \left( \frac{1}{m} \right) + \frac{1}{m} \sum_{l=1}^{m-1} l \cdot \log |1 - \zeta_m^l| \right) + (z-1) \cdot (\dots). \end{aligned}$$

In particular, if  $x = 1$  then  $R_x$  is the zero function. Otherwise,  $R_x$  has a meromorphic continuation to the entire complex plane, with poles precisely at the finite set  $\{x^n \mid n \in \mathbf{Z}\}$ , all having order 1, and the pole at  $z = 1$  has residue  $\frac{1}{m} \log \left( \frac{1}{m} \right)$ .

*Remark 2.6.* This formulation of the proposition is best for our purposes, but there is also a different perspective relating this to special  $L$ -values, see Prop. 9.1.

*Proof.* Suppose  $x$  is a primitive  $m$ -th root of unity. We write  $x = \zeta_m$ . If  $m = 1$ , the function  $R_x$  is zero by definition, so we may assume  $m \geq 2$ . Then

$$\begin{aligned} R_x(z) &= \sum'_{r \geq 1} \log |1 - \zeta_m^r| \cdot z^r = \sum_{n \geq 0} \sum_{l=1}^{m-1} \log |1 - \zeta_m^{mn+l}| \cdot z^{mn+l} \\ &= \sum_{l=1}^{m-1} \log |1 - \zeta_m^l| \frac{z^l}{1 - z^m}. \end{aligned}$$

Now,

$$\begin{aligned}
\frac{z^l}{1-z^m} &= \frac{((z-1)+1)^l}{1-((z-1)+1)^m} = \frac{\sum_{t=0}^l \binom{l}{t} (z-1)^t}{1 - \sum_{k=0}^m \binom{m}{k} (z-1)^k} \\
&= -\frac{1}{m} \frac{1}{z-1} \left( \sum_{t=0}^l \binom{l}{t} (z-1)^t \right) \left( \sum_{r=0}^{\infty} \left( -\frac{1}{m} \sum_{k=2}^m \binom{m}{k} (z-1)^{k-1} \right)^r \right) \\
&= -\frac{1}{m} \frac{1}{z-1} - \frac{l}{m} + \frac{m-1}{2m} + (z-1)(\dots).
\end{aligned}$$

Thus, expanding  $R_x$  at  $z = 1$  yields

$$\begin{aligned}
&= \sum_{l=1}^{m-1} \log |1 - \zeta_m^l| \left( -\frac{1}{m} \frac{1}{z-1} - \frac{l}{m} + \frac{m-1}{2m} \right) + (z-1)(\dots) \\
&= -\frac{1}{m} \frac{1}{z-1} \sum_{l=1}^{m-1} \log |1 - \zeta_m^l| + \frac{m-1}{2m} \sum_{l=1}^{m-1} \log |1 - \zeta_m^l| \\
&\quad - \frac{1}{m} \sum_{l=1}^{m-1} l \cdot \log |1 - \zeta_m^l| + (z-1) \cdot (\dots).
\end{aligned}$$

Finally,  $\sum_{l=1}^{m-1} \log |1 - \zeta_m^l| = \log \left| \prod_{l=1}^{m-1} (1 - \zeta_m^l) \right| = \log(m)$  for  $m \geq 2$ , so

$$= \frac{1}{m} \log \left( \frac{1}{m} \right) \frac{1}{z-1} - \frac{m-1}{2} \frac{1}{m} \log \left( \frac{1}{m} \right) - \frac{1}{m} \sum_{l=1}^{m-1} l \cdot \log |1 - \zeta_m^l| + (z-1) \cdot (\dots),$$

finishing the proof.  $\square$

### 3. ANALYTIC CONTINUATION OF $R_x$ FOR $|x| > 1$

The next ‘easy’ case is  $R_x$  for  $|x| > 1$ , because it can be reduced to the case  $|x| < 1$ :

**Lemma 3.1.** *If  $|x| > 1$ , then the series  $R_x$  converges inside the unit disc and inside this disc, we have the identity*

$$R_x(z) = R_{x^{-1}}(z) + \log |x| \cdot \frac{z}{(z-1)^2}.$$

*Proof.* If  $|x| > 1$ , we have  $|x^{-1}| < 1$  and so below the left-hand side converges in the unit disc,  $R_{x^{-1}}(z)$  equals

$$\begin{aligned}
&= \sum_{r \geq 1} \log |1 - x^{-r}| z^r = \sum_{r \geq 1} \log (|x^{-r}| |x^r - 1|) z^r \\
&= \sum_{r \geq 1} \log (|x^{-r}|) z^r + \sum_{r \geq 1} \log |1 - x^r| z^r = -\log |x| \sum_{r \geq 1} r z^r + \sum_{r \geq 1} \log |1 - x^r| z^r,
\end{aligned}$$

which is exactly the claim that we wished to prove.  $\square$

Next, we want to determine the first terms of the expansion of  $R_x(z)$  around  $z = 1$ . To this end, let  $p$  be the partition function, i.e.  $p(n)$  is the number of distinct presentations of  $n$  as a sum of integers  $\geq 1$ , irrespective of the order. Let  $F$  be its generating function, i.e.

$$(3.1) \quad F(z) = \sum_{n \geq 0} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}.$$



**Lemma 3.2.** *Suppose  $x \in \mathbf{C}^\times$  with  $|x| < 1$ . Then the expansion of  $R_x(z)$  around  $z = 1$  is given by*

$$R_x(z) = -\log |F(x)| + \sum_{\ell=0}^{\infty} (z-1)^{\ell+1} \left( \sum_{r \geq 1} \log |x^r - 1| \binom{r}{\ell+1} \right)$$

and all the sums in the round brackets on the right converge.

*Proof.* The convergence of the sums is harmless: By Lemma 2.3, since  $|x| < 1$ , we have

$$\left| \log |x^r - 1| \binom{r}{\ell+1} \right| \leq 2 |x|^r \binom{r}{\ell+1}$$

and then  $\sum_{r \geq 1} \log |x^r - 1| \binom{r}{\ell+1}$  is dominated by  $2 \sum_{r \geq 1} |x|^r \binom{r}{\ell+1}$  and the latter is the standard estimate for convergence of a power series. In particular, since  $|x| < 1$ , this converges. We compute

$$R_x(z) = \sum_{r \geq 1} \log |1 - x^r| z^r = \sum_{r \geq 1} \log |1 - x^r| (z^r - 1) + \sum_{r \geq 1} \log |1 - x^r|,$$

but in view of Equation 3.1 the rightmost term is just a special value of the generating function of the partition function. Thus,

$$\begin{aligned} &= -\log |F(x)| + \sum_{r \geq 1} \log |x^r - 1| \cdot (z-1)(1 + z + \cdots + z^{r-1}) \\ &= -\log |F(x)| + \sum_{r \geq 1} \log |x^r - 1| \cdot (z-1) \sum_{a=0}^{r-1} ((z-1) + 1)^a \end{aligned}$$

and expanding this using the binomial formula yields

$$= -\log |F(x)| + \sum_{\ell=0}^{\infty} (z-1)^{\ell+1} \left( \sum_{r \geq 1} \log |x^r - 1| \cdot \sum_{a=0}^{r-1} \binom{a}{\ell} \right).$$

By a standard identity on binomial coefficients, the innermost sum equals  $\binom{r}{\ell+1}$ , confirming our claim.  $\square$

*Aside 3.3.* The modular discriminant  $\Delta$  is a weight 12 modular form for  $\mathrm{SL}_2(\mathbf{Z})$  and given by

$$\Delta(\tau) = q \prod_{m \geq 1} (1 - q^m)^{24} \quad \text{for} \quad q = e^{2\pi i \tau}, \tau \in \mathbf{H}.$$

For its logarithm, we obtain  $\frac{1}{24} (\log \Delta(\tau) - \log q) = \sum_{m \geq 1} \log(1 - q^m)$  and thus for  $F(x) \in \mathbf{C} \setminus \mathbf{R}_{\leq 0}$ , we get

$$-\log |F(x)| = \sum_{n=1}^{\infty} \log(1 - x^n) + \sum_{n=1}^{\infty} \log(1 - \bar{x}^n).$$

Thus, we may spell out  $-\log |F(x)|$  as an expression in the function  $\log \Delta$ . For  $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ , the transformation behaviour of  $\log \Delta$  has an explicit description in terms of Dedekind symbols/sums, [RG72, Ch. 4]. This gives us a rich structure on the constant coefficient of the Laurent expansion of  $R_x$  at  $z = 1$ . I have been wondering whether this structure would give rise to some visible patterns in the behaviour of  $R_x$  when changing  $x$ , but I have not been able to isolate anything meaningful. Maybe somebody else has an idea.

4. ANALYTIC CONTINUATION OF  $R_x$  FOR  $|x| < 1$ 

**4.1. Choice for complex exponentiation.** We shall mostly work with the principal branch of the logarithm. For us, this means that it is defined on  $\mathbf{C}^\times$  and given by

$$(4.1) \quad \log(re^{i\theta}) = \log r + i\theta \quad \text{for } \theta \in (-\pi, +\pi].$$

Based on this choice of a logarithm,  $x^s := \exp(s \cdot \log x)$  is our choice of the meaning of complex exponentiation. We use capital letters “Log” whenever we want to stress that an arbitrary branch of the logarithm can be used.

*Remark 4.1.* Usually, both  $x$  and  $s$  will be complex numbers, so we will have to be very careful with deceptive functional equations, e.g.  $e^{st} \neq (e^s)^t$  for general  $s, t \in \mathbf{C}$ .

**Lemma 4.2.** *For  $x \in \mathbf{C}^\times$  and  $s \in \mathbf{C}$ , one has  $|x^s| = |x|^{\operatorname{Re} s} \cdot e^{-\arg(x) \operatorname{Im} s}$ .*

*Proof.* We have, for  $x = |x|e^{i\theta}$ ,  $\theta := \arg x \in (-\pi, \pi]$  and  $s = a + bi$  with  $a, b \in \mathbf{R}$ ,

$$\begin{aligned} x^s &= e^{s \log x} = e^{(a+bi)(\log|x| + i\theta)} = e^{(a \log|x| - b\theta) + i(a\theta + b \log|x|)} \\ &= e^{a \log|x|} \cdot e^{-b\theta} \cdot e^{i(a\theta + b \log|x|)} = |x|^{\operatorname{Re} s} \cdot e^{-\arg(x) \operatorname{Im} s} \cdot e^{i(a\theta + b \log|x|)} \end{aligned}$$

and taking absolute values, we get the claim.  $\square$

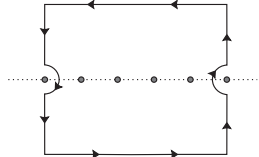
**4.2. Integral forms.**

**Proposition 4.3.** *Let  $K > 0$  be real. Suppose  $h : [1, +\infty) \times i[-K, K] \rightarrow \mathbf{C}$  is a function which admits a holomorphic continuation to an open neighbourhood of this box. Then for all integers  $1 \leq a < b$ , we have*

$$\begin{aligned} \sum_{n=a}^b h(n) &= \frac{1}{2}h(a) + \frac{1}{2}h(b) + \int_a^b h(s)ds \\ &\quad + i \int_0^K \frac{h(a+iy) - h(a-iy)}{e^{2\pi y} - 1} dy - i \int_0^K \frac{h(b+iy) - h(b-iy)}{e^{2\pi y} - 1} dy \\ &\quad - \int_{a+iK}^{b+iK} \frac{h(s)}{1 - e^{-2\pi is}} ds + \int_{a-iK}^{b-iK} \frac{h(s)}{e^{2\pi is} - 1} ds. \end{aligned}$$

This is a modification of the Abel–Plana contour. A modern reference is Olver’s book [Olv97, Ch. 8, §3]. However, our version is a little more complicated as we cannot let  $K$  go to infinity.

*Proof.* We begin with the cotangent series, spelled out below, which is compactly convergent in  $\mathbf{C} \setminus \mathbf{Z}$ . We may consider the positively oriented contour  $\mathbf{C}$  around the set  $\text{Box} := [a, b] \times i[-K, K]$ , and slightly modify it at  $a$  and  $b$  by cutting out little semi-circles of radius  $\delta > 0$ , chosen sufficiently small, say  $< \frac{1}{4}K$ , as in the following figure:



$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} + \frac{1}{z-n} \right),$$

The grey dots represent the integers  $\mathbf{Z}$ , that is the poles of the cotangent and we need these semi-circles to avoid passing right through a pole. If  $\mathbf{H}$  denotes the upper half-plane, the

Residue Theorem gives us

$$(4.2) \quad \sum_{n=a+1}^{b-1} h(n) - \int_{a+\delta}^{b-\delta} h(z) dz = \frac{1}{2i} \int_{\mathbf{C}} h(z) \cot(\pi z) dz \\ + \frac{1}{2} \int_{\mathbf{H} \cap \mathbf{C}} h(z) dz - \frac{1}{2} \int_{-\mathbf{H} \cap \mathbf{C}} h(z) dz.$$

The second line arises from writing the integral along  $[a, b]$  either by going through the upper half of the contour, or the lower half, and since both work equally well, we can just as well take the average of these two (equal) values. As is well-known, we have  $\frac{1}{i} \cot(z) = \frac{1}{i} \frac{\cos(z)}{\sin(z)} = \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = \frac{e^{2iz} + 1}{e^{2iz} - 1}$ , which may be simplified to either of the following two expressions  $1 + 2\frac{1}{e^{2iz} - 1}$  or  $2\frac{1}{1 - e^{-2iz}} - 1$ , whichever is more convenient. We subdivide the contour  $\mathbf{C}$  of the first integral into its upper and lower part as well. This yields

$$= \frac{1}{2i} \int_{\mathbf{H} \cap \mathbf{C}} h(z) \cot(\pi z) dz + \frac{1}{2i} \int_{-\mathbf{H} \cap \mathbf{C}} h(z) \cot(\pi z) dz \\ + \frac{1}{2} \int_{\mathbf{H} \cap \mathbf{C}} h(z) dz - \frac{1}{2} \int_{-\mathbf{H} \cap \mathbf{C}} h(z) dz$$

and then, using the aforementioned two presentations, this can be rewritten as

$$= \int_{\mathbf{H} \cap \mathbf{C}} h(z) \left( \frac{1}{1 - e^{-2\pi iz}} - \frac{1}{2} \right) dz + \int_{-\mathbf{H} \cap \mathbf{C}} h(z) \left( \frac{1}{2} + \frac{1}{e^{2\pi iz} - 1} \right) dz \\ + \frac{1}{2} \int_{\mathbf{H} \cap \mathbf{C}} h(z) dz - \frac{1}{2} \int_{-\mathbf{H} \cap \mathbf{C}} h(z) dz = \int_{\mathbf{H} \cap \mathbf{C}} \frac{h(z)}{1 - e^{-2\pi iz}} dz + \int_{-\mathbf{H} \cap \mathbf{C}} \frac{h(z)}{e^{2\pi iz} - 1} dz,$$

involving a convenient cancellation of terms. Firstly, one checks using the continuity of  $h$  that for  $\delta \rightarrow 0$  one gets additional terms of  $-\frac{1}{2}h(a)$  and  $-\frac{1}{2}h(b)$ , so adding the summands for  $n = a, b$  to the left-hand side of Equation 4.2 (so that the sum now reads  $\sum_a^b$ ), we need to add  $\frac{1}{2}h(a) + \frac{1}{2}h(b)$  on the right-hand side to keep it balanced. Replace  $\mathbf{C}$  by a straight left and right edge with a tiny omission of radius  $\delta > 0$  instead of the semi-circles, and this will be justified later as we shall see that the remaining limit will exist for  $\delta \rightarrow 0$ . For the left edge, pick the curve  $z(y) := a + iy$  and use that  $e^{\pm 2\pi i(a+iy)} = e^{\mp 2\pi y}$  because  $a \in \mathbf{Z}$ . Thus,

$$(4.3) \quad \int_{\mathbf{H} \cap \mathbf{C}_{left}} \frac{h(z)}{1 - e^{-2\pi iz}} dz = -i \int_{\delta}^K \frac{h(a + iy)}{1 - e^{2\pi y}} dy \\ \int_{-\mathbf{H} \cap \mathbf{C}_{left}} \frac{h(z)}{e^{2\pi iz} - 1} dz = -i \int_{\delta}^K \frac{h(a - iy)}{e^{2\pi y} - 1} dy.$$

The same works for the right edge with  $b$  in place of  $a$ . Finally, take the limit  $\delta \rightarrow 0$ . This is harmless: Both numerator and denominator are zero for  $y = 0$ , but by L'Hôpital's rule, the limit of the integrand for  $y \rightarrow 0$  agrees with  $\lim_{y \rightarrow 0} \frac{i}{2\pi} (h'(b + iy) + h'(b - iy)) e^{-2\pi y} = \frac{i}{\pi} h'(b)$ , and in particular this limit exists.  $\square$

Below, we shall repeatedly need a case distinction “ $\pm$ ”. Either case comes with an assumption, which we shall repeatedly need, so we give it a name:

**Assumption ( $\mathfrak{C}^\pm$ ).** Suppose  $x \in \mathbf{C}^\times$  and  $|x| < 1$  is chosen. If  $\mp \arg(x) > 0$ , choose a real number  $K$  such that

$$0 < K < -\frac{\log|x|}{\mp \arg x}.$$

If  $\mp \arg(x) \leq 0$ , we only assume  $K > 0$  and do not impose an upper bound.

**Lemma 4.4.** Fix a case “ $\pm$ ”. If Assumption  $\mathfrak{C}^\pm$  is satisfied, then for all  $s = u \pm iv$  with  $u \geq 1$  and  $0 \leq v < K$ , we have  $|x^s| < 1$ . If  $u > 1$  and  $0 \leq v \leq K$ , we also have  $|x^s| < 1$ .

*Proof.* We have  $|x^s| = |e^{s \log x}| = |e^{(u \pm iv)(\log|x| + i \arg x)}| = |e^{u \log|x| \mp v \arg x}| = |x|^u \cdot e^{\mp(\arg x)v}$ . If  $\mp \arg(x) \leq 0$ , we surely have  $|e^{\mp(\arg x)v}| \leq 1$  and  $|x|^u < 1$  by  $u \geq 1$  and  $|x| < 1$ , proving the claim in this case. Otherwise, if  $\mp \arg(x) > 0$ , we have  $v < K < -\frac{\log|x|}{\mp \arg x}$  (resp. equality) and thus  $|x|^u \cdot e^{\mp(\arg x)v} < |x|^{u-1}$  and for  $u \geq 1$  (resp.  $> 1$ ) the claim follows from  $|x| < 1$ .  $\square$

**Proposition 4.5.** Suppose  $x \in \mathbf{C}^\times$  and  $|x| < 1$ . Then for all  $\operatorname{Re} w > 0$ , the integral

$$A_{w,x}(w) := \int_1^\infty \log(1 - x^s) e^{-ws} ds$$

converges and defines a holomorphic function on the right half-plane. The series

$$\tilde{A}_{w,x}(w) := \sum_{n \geq 1} \frac{1}{n} \frac{e^{n \log x - w}}{n \log x - w}$$

is compactly convergent everywhere outside  $\mathbf{Z}_{\geq 1} \cdot \log x$  in the complex plane and defines a meromorphic continuation with poles at these points.

Note that all these poles lie in the open left half-plane.

*Proof. (Step 1)* For real numbers  $s \geq 1$ , we have  $|x^s| = |x|^s < 1$ , so inside the range of integration,  $\log(1 - x^s)$  can be expanded as a uniformly convergent power series. Concretely, for  $1 \leq a < b$  we get

$$\int_a^b \log(1 - x^s) e^{-ws} ds = - \int_a^b \sum_{n \geq 1} \frac{x^{sn}}{n} e^{-ws} ds$$

and swapping integral and sum, this can be integrated in an explicit fashion:

$$\begin{aligned} &= - \sum_{n \geq 1} \frac{1}{n} \int_a^b x^{sn} e^{-ws} ds = - \sum_{n \geq 1} \frac{1}{n} \int_a^b e^{(n \log x - w)s} ds \\ &= - \sum_{n \geq 1} \frac{1}{n} \begin{cases} \frac{e^{(n \log x - w)s}}{n \log x - w} \Big|_{s=a}^{s=b} & \text{for } n \log x - w \neq 0 \\ b - a & \text{otherwise.} \end{cases} \end{aligned}$$

*(Step 2)* Now, assume  $\operatorname{Re} w > 0$  and consider the case  $b \rightarrow +\infty$ . Then thanks to  $\operatorname{Re} w > 0$ , we have  $e^{-bw} \rightarrow 0$ , and so  $|e^{n \log x - w}|^b = |x^n|^b e^{-bw} \rightarrow 0$ . Thus, *Step 1* implies that  $A_{w,x} = \tilde{A}_{w,x}$  in the right half-plane. However,  $\tilde{A}_{w,x}$  is uniformly convergent in any compactum outside the poles, even without assuming  $\operatorname{Re} w > 0$ . To see this, note that  $|e^{n \log x - w}| = |x|^n \cdot e^{-\operatorname{Re} w}$  and since the denominator can be bounded in any compactum outside the poles, the convergence is dominated by a convergent geometric series because of  $|x| < 1$ . Thus,  $\tilde{A}_{w,x}$  is meromorphic in the entire complex plane.  $\square$

**Proposition 4.6.** *Suppose we are in the situation of Assumption  $\mathfrak{C}^\pm$ . Let  $a \geq 1$  be an integer and  $0 < \delta < K$ . Then for all  $\operatorname{Re} w > 0$ , the integral*

$$M_{a,K}^\pm(w) := \int_\delta^K \frac{\log(1 - x^{a \pm iy}) e^{-w(a \pm iy)}}{e^{2\pi y} - 1} dy$$

*converges and defines a holomorphic function. The series*

$$\tilde{M}_{a,K}^\pm(w) := -e^{-wa} \sum_{m,n \geq 1} \frac{x^{ma}}{m} \begin{cases} \frac{e^{(\pm im \log x \mp iw - 2\pi n)y}}{\pm im \log x \mp iw - 2\pi n} \Big|_{y=\delta}^{y=K} & \text{for } \pm im \log x \mp iw - 2\pi n \neq 0 \\ K - \delta & \text{otherwise.} \end{cases}$$

*is compactly convergent in the entire complex plane and defines a holomorphic continuation of  $M_a^\pm$ .*

*Proof. (Step 1)* We assume  $\operatorname{Re} w > 0$ . Then the integral

$$M_{a,K}^\pm(w) := \int_\delta^K \frac{\log(1 - x^{a \pm iy}) e^{-w(a \pm iy)}}{e^{2\pi y} - 1} dy = \int_\delta^K \frac{\log(1 - x^{a \pm iy}) e^{-w(a \pm iy)}}{e^{2\pi y} (1 - e^{-2\pi y})} dy.$$

is convergent. Since  $y \geq \delta > 0$  within the range of integration, we have  $|e^{-2\pi y}| < 1$  and we may expand  $1/(1 - e^{-2\pi y})$  as a uniformly convergent geometric series. We obtain

$$= \int_\delta^K \log(1 - x^{a \pm iy}) e^{-w(a \pm iy)} \sum_{n \geq 1} e^{-2\pi ny} dy.$$

By Assumption  $\mathfrak{C}^\pm$  and Lemma 4.4, we have  $|x^{a \pm iy}| < 1$  since  $a \geq 1$  and  $0 \leq y < K$ . Thus, again we may expand the logarithm and we obtain

$$\begin{aligned} &= - \int_\delta^K \sum_{m \geq 1} \frac{x^{(a \pm iy)m}}{m} e^{-w(a \pm iy)} \sum_{n \geq 1} e^{-2\pi ny} dy \\ &= - \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \int_\delta^K e^{am \log x} e^{\pm iym \log x} e^{-w(a \pm iy)} e^{-2\pi ny} dy \end{aligned}$$

by uniform convergence. Then,

$$\begin{aligned} &= -e^{-wa} \sum_{m,n \geq 1} \frac{x^{ma}}{m} \int_\delta^K e^{(\pm im \log x \mp iw - 2\pi n)y} dy \\ (4.4) \quad &= -e^{-wa} \sum_{m,n \geq 1} \frac{x^{ma}}{m} \begin{cases} \frac{e^{(\pm im \log x \mp iw - 2\pi n)y}}{\pm im \log x \mp iw - 2\pi n} \Big|_{y=\delta}^{y=K} & \text{for } \pm im \log x \mp iw - 2\pi n \neq 0 \\ K - \delta & \text{otherwise.} \end{cases} \end{aligned}$$

(Step 2) Next, we claim that this last expression converges for all  $w \in \mathbf{C}$ , dropping the assumption  $\operatorname{Re} w > 0$ . To this end, we compute

$$X_{n,m} := |x^{ma}| \cdot \left| e^{(\pm im \log x \mp iw - 2\pi n)y} \right| = \left| e^{m(a \log x \pm iy \log x)} \right| \cdot |e^{\mp iwy}| \cdot |e^{-2\pi y}|^n$$

Now, since  $y > 0$ , we have  $|e^{-2\pi y}|^n < 1$  for all  $n$ . The term  $|e^{\mp iwy}|$  does not depend on  $n$  nor  $m$ . Finally,

$$\left| e^{m(a \log x \pm iy \log x)} \right| = \left| e^{m(a(\log|x| + i \arg x) \pm iy(\log|x| + i \arg x))} \right| = \left| e^{(a \log|x| \mp y \arg x)} \right|^m$$

If  $\mp \arg(x) \leq 0$ , then  $\mp y \arg x \leq 0$  and by  $a \geq 1$  and  $|x| < 1$ , it follows that  $e$  has a negative exponent. Thus, we get a term of the shape  $\theta^m$  for some  $0 < \theta < 1$ . Now: Since  $|e^{-2\pi y}| < 1$  and  $\theta < 1$ , the terms  $X_{n,m}$  are dominated by a geometric series both in the

variables  $n$  and  $m$ . It is easy to check that the apparent poles in the top case of Equation 4.4 are all removable, and in fact the holomorphic continuation is given by switching to the second case. In particular, we get an everywhere compactly convergent series of holomorphic functions.  $\square$

**Proposition 4.7.** *Suppose we are in the situation of Assumption  $\mathfrak{C}^\pm$ . Let  $a \geq 1$  be an integer. Then for all  $\operatorname{Re} w > 0$ , the integral*

$$T_K^+(w) := - \int_{1+iK}^{\infty+iK} \frac{\log(1-x^s)e^{-ws}}{1-e^{-2\pi is}} ds$$

*resp.*

$$T_K^-(w) := + \int_{1-iK}^{\infty-iK} \frac{\log(1-x^s)e^{-ws}}{e^{2\pi is} - 1} ds$$

*converges and defines a holomorphic function in the open right half-plane. The series*

$$(4.5) \quad \tilde{T}_K^\pm(w) = \pm \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \frac{e^{(m \log x - w \pm 2\pi in)(1 \pm iK)}}{m \log x - w \pm 2\pi in}$$

*is compactly convergent everywhere outside  $\mathbf{Z}_{\geq 1} \cdot \log x \pm \mathbf{Z}_{\geq 1} \cdot 2\pi i$  in the entire complex plane. It defines a meromorphic continuation with poles at the said points.*

Here the integration from “ $1 \pm iK$ ” to “ $\infty \pm iK$ ” is meant to denote integration along the curve  $\gamma(t) := t \pm iK$  for  $t \in [1, +\infty)$ .

*Proof.* (Step 1) For  $T^+$  we get

$$- \int_{a+iK}^{b+iK} \frac{\log(1-x^s)e^{-ws}}{1-e^{-2\pi is}} ds.$$

We compute

$$- \int_{a+iK}^{b+iK} \frac{\log(1-x^s)e^{-ws}}{1-e^{-2\pi is}} ds = - \int_{a+iK}^{b+iK} \frac{\log(1-x^s)e^{-ws}}{e^{-2\pi is}(e^{2\pi is} - 1)} ds$$

and since we are in the upper half-plane  $|e^{2\pi is}| = e^{-2\pi K} < 1$  by  $K > 0$ . Thus, we may expand this as a geometric series which is uniformly convergent in the range of integration,

$$= \int_{a+iK}^{b+iK} \log(1-x^s)e^{-ws} \sum_{n \geq 1} e^{2\pi ins} ds.$$

The case of  $T^-$  is very much analogous: We get

$$- \int_{a-iK}^{b-iK} \log(1-x^s)e^{-ws} \sum_{n \geq 1} e^{-2\pi ins} ds$$

instead. Thus, up to signs, we may handle both cases simultaneously. By Assumption  $\mathfrak{C}^\pm$ ,  $1 \leq a < b$  and Lemma 4.4, we can expand the logarithm,

$$\begin{aligned} &= \mp \int_{a \pm iK}^{b \pm iK} \sum_{m \geq 1} \frac{x^{sm}}{m} e^{-ws} \sum_{n \geq 1} e^{\pm 2\pi ins} ds = \mp \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \int_{a \pm iK}^{b \pm iK} e^{(m \log x - w \pm 2\pi in)s} ds \\ &= \mp \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \begin{cases} \frac{e^{(m \log x - w \pm 2\pi in)s}}{m \log x - w \pm 2\pi in} \Big|_{s=a \pm iK}^{s=b \pm iK} & \text{for } m \log x - w \pm 2\pi in \neq 0 \\ b - a & \text{otherwise.} \end{cases} \end{aligned}$$

(Step 2) We will estimate the magnitude of the integrand/summands. Suppose  $s = u \pm iv$  with  $u, v \in \mathbf{R}$ ,  $v \geq 0$  (it makes sense to assume  $v \geq 0$  because for handling  $T^+$  it suffices to have estimates which are valid in the upper half-plane, and correspondingly for  $T^-$  in the lower half-plane). Now,

$$\left| e^{(m \log x - w \pm 2\pi i n)s} \right| = \left| e^{mu \log |x| \mp mv \arg(x)} \right| \cdot |e^{-ws}| \cdot |e^{-2\pi v}|^n.$$

For all  $s$  in the range of integration, the imaginary part is  $\pm K$ , so  $|e^{-2\pi K}| < 1$  by  $K > 0$ . It follows that our terms are dominated by a geometric series in  $n$ . Moreover,  $|e^{mu \log |x| \mp mv \arg(x)}| = |e^{u \log |x| \mp v \arg(x)}|^m$  and by Assumption  $\mathfrak{C}^\pm$  we have: In the range of integration,  $u \geq 1$  and  $\log |x|$  is negative, so if  $\mp \arg(x) \leq 0$ , the exponent is negative and thus we also have domination by a geometric series in  $m$ . If  $\mp \arg(x) > 0$  on the other hand, we have

$$v < K < -\frac{\log |x|}{\mp \arg x}$$

by Assumption  $\mathfrak{C}^\pm$ . Thus,  $\log |x| \mp v \arg x < 0$ , and since  $\log |x| < 0$  and  $u \geq 1$ , adding  $(u-1) \log |x| \leq 0$  yields

$$u \log |x| \mp v \arg x < (u-1) \log |x| \leq 0.$$

Thus, again the exponent is negative, giving domination by a geometric series also in  $m$ . Finally, note that for  $w = c + di$ , the term  $|e^{-ws}|$  can be evaluated to be

$$|e^{-ws}| = \left| e^{-(c+di)u \mp iv(c+di)} \right| = |e^{-cu}| \cdot |e^{\pm vd}|.$$

So, if in Step 1 we let  $b \rightarrow +\infty$ , and for  $c > 0$ ,  $|e^{-cu}|$  goes to zero. As this universally bounds all coefficients, we see that the right-hand side boundary term of the integration vanishes if we assume  $\operatorname{Re} w > 0$  (i.e.  $c > 0$ ). Moreover, our upper bound of exponential decay in both  $n$  and  $m$  shows that outside the poles, we have uniform convergence of the series in line 4.5 in any compactum.  $\square$

**Proposition 4.8.** *Suppose  $x \in \mathbf{C}^\times$  and  $|x| < 1$ . Then for all  $\operatorname{Re} w > 0$ , the series*

$$Q_x(w) := \sum_{n=1}^{\infty} \log(1-x^n) e^{-wn}$$

*is compactly convergent and defines a holomorphic function in the right half-plane. It admits a meromorphic continuation to the entire complex plane with poles at  $\mathbf{Z}_{\geq 1} \cdot \log x + \mathbf{Z} \cdot 2\pi i$ .*

*Proof.* Firstly, we choose some  $K$  such that the Assumptions  $\mathfrak{C}^+$  and  $\mathfrak{C}^-$  are both met. This is always possible: If  $\mp \arg(x) \leq 0$ , we only need  $K > 0$ , and if  $\mp \arg(x) > 0$ , note that the right-hand side in

$$0 < K < -\frac{\log |x|}{\mp \arg x}$$

is strictly positive since  $|x| < 1$ . Thus, some  $K$  in between these bounds exists and we fix a choice. Assume  $\operatorname{Re} w > 0$ . Now we apply Prop. 4.3 for  $b \rightarrow +\infty$ . Easy estimates show that  $h(b) \rightarrow 0$  and the right edge term vanishes as well, and by combining the previous

propositions, we get

$$\begin{aligned} \sum_{n=1}^{\infty} \log(1-x^n)e^{-wn} &= \frac{1}{2} \log(1-x)e^{-w} + \sum_{n \geq 1} \frac{1}{n} \frac{e^{n \log x - w}}{n \log x - w} + i(\tilde{M}_{1,K}^+(w) - \tilde{M}_{1,K}^-(w)) \\ &\quad - \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \frac{e^{(m \log x - w + 2\pi i n)(1+iK)}}{m \log x - w + 2\pi i n} - \sum_{m \geq 1} \sum_{n \geq 1} \frac{1}{m} \frac{e^{(m \log x - w - 2\pi i n)(1-iK)}}{m \log x - w - 2\pi i n}. \end{aligned}$$

This identity holds only for  $\operatorname{Re} w > 0$ , but the right-hand side is a meromorphic function in the entire complex plane thanks to the quoted propositions. Its poles are located at

$$\{\mathbf{Z}_{\geq 1} \cdot \log x\} \cup \{\mathbf{Z}_{\geq 1} \cdot \log x + \mathbf{Z}_{\neq 0} \cdot 2\pi i\} = \mathbf{Z}_{\geq 1} \cdot \log x + \mathbf{Z} \cdot 2\pi i,$$

since  $\tilde{M}_1^{\pm}$  is holomorphic in the entire complex plane. Finally, connectedness of  $\mathbf{C}$  minus these poles and the identity principle imply that our choice of  $K$  does not affect the continuation.  $\square$

*Remark 4.9.* It is not surprising that the pole locus is  $2\pi i$ -periodic, since  $Q_w$  is clearly periodic under  $w \mapsto w + 2\pi i$ . Note that most of the summands that we had individually analytically continued do not enjoy such a periodicity by themselves. Only their sum is periodic.

Now we are ready to prove our first key ingredient for the analytic continuation.

**Theorem 4.10.** *Suppose  $x \in \mathbf{C}^{\times}$  and  $|x| < 1$ .*

- (1) *Then the series  $R_x(z)$  admits a meromorphic continuation to the entire complex plane with poles at  $\{x^{\mathbf{Z}_{\leq -1}}, \bar{x}^{\mathbf{Z}_{\leq -1}}\}$ . These poles have order 1.*
- (2) *In explicit terms, pick any sufficiently small choice of some  $K > 0$ . Then for all  $z \in \mathbf{C}$  outside this set of poles, this continuation is given by*

$$R_x(z) = \frac{1}{2} (\tilde{q}_x(z) + \tilde{q}_{\bar{x}}(z)),$$

where

$$\begin{aligned} \tilde{q}_x(z) &:= \frac{1}{2} \log(1-x)z + z \sum_{n \geq 1} \frac{1}{n} \frac{x^n}{n \log x + \operatorname{Log} z} + i(\tilde{M}_{1,K}^+(-\operatorname{Log} z) - \tilde{M}_{1,K}^-(-\operatorname{Log} z)) \\ &\quad - \sum_{m,n \geq 1} \frac{1}{m} \frac{e^{(m \log x + \operatorname{Log} z + 2\pi i n)(1+iK)}}{m \log x + \operatorname{Log} z + 2\pi i n} - \sum_{m,n \geq 1} \frac{1}{m} \frac{e^{(m \log x + \operatorname{Log} z - 2\pi i n)(1-iK)}}{m \log x + \operatorname{Log} z - 2\pi i n}, \end{aligned}$$

with  $\tilde{M}_{1,K}^{\pm}$  as in Prop. 4.6, and  $\operatorname{Log} z$  is any choice of a logarithm defined in a neighbourhood of  $z$ . In particular, the value of  $\tilde{q}_x$  is independent of this choice and the choice of  $K$ .

*Proof.* We use a trick: (*Trick*) Firstly, define  $\tilde{h}_{w,x}(s) := \log(1-x^n)e^{-ws}$ . Then for all integers  $n$ , we have

$$\begin{aligned} \tilde{h}_{w,x}(n) + \tilde{h}_{w,\bar{x}}(n) &= \log(1-e^{n \log x})e^{-wn} + \log(1-e^{n \log \bar{x}})e^{-wn} \\ &= \left( \log(1-e^{n \log x}) + \log(\overline{1-e^{n \log x}}) \right) e^{-wn} = 2 \log |1-e^{n \log x}| \cdot e^{-wn}. \end{aligned}$$

The key point that we have used is that we only need this formula for  $n \in \mathbf{N}$ , in particular  $n$  is real. Thus,  $\bar{n} = n$ , which would be false for a general  $s$ . Moreover, we have used the identities  $\log z + \log \bar{z} = 2 \log |z|$  and  $\log \bar{z} = \overline{\log z}$ , which follow from our choice of the



logarithm, Equation 4.1, and which need *not* hold for other branches. For all  $w \in \mathbf{C}$  with  $\operatorname{Re} w > 0$ , we have  $|e^{-w}| < 1$ , so by Lemma 2.4 the series

$$\begin{aligned} R_x(e^{-w}) &= \sum_{n=1}^{\infty} \log|1 - x^n| \cdot e^{-wn} = \frac{1}{2} \sum_{n=1}^{\infty} \log(1 - x^n) \cdot e^{-wn} \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} \log(1 - \bar{x}^n) \cdot e^{-wn} = \frac{1}{2} (Q_x(w) + Q_{\bar{x}}(w)) \end{aligned}$$

is uniformly convergent and defines a holomorphic function in the right half-plane. Thanks to Prop. 4.8,  $w \mapsto R_x(e^{-w})$  has a meromorphic continuation to the entire complex plane, call it  $\widetilde{RW}_x(w)$ .

(*Conclusion*) This analytic continuation must be periodic under  $w \rightarrow w + 2\pi i$ . To see this, note that it is true for  $R_x(e^{-w})$  in the right half-plane, simply since it is true for  $e^{-w}$ . Thus,  $\widetilde{RW}_x$  must also be periodic. Let  $\operatorname{Log} : U \rightarrow \mathbf{C}$  be some branch of the logarithm, defined on some domain  $U \subset \mathbf{C}$ . Define a function

$$\tilde{R}_x : U \rightarrow \mathbf{C}, \quad \tilde{R}_x(z) := \widetilde{RW}_x(-\operatorname{Log} z).$$

This makes  $\tilde{R}_x$  a meromorphic function on  $U$ . Since all branches of the logarithm differ by multiples of  $2\pi i$ , and  $\widetilde{RW}_x$  is  $2\pi i$ -periodic, it follows that choosing different  $U$  and different branches, the definitions of  $\tilde{R}_x$  on the various opens  $U$  glue. This defines a meromorphic function on the entire complex plane. Finally, we observe that  $\tilde{R}_x(e^{-w}) = R_x(e^{-w})$  for  $\operatorname{Re} w > 0$ , so this is an analytic continuation. The poles of  $Q_x$  become poles at  $\{x^{\mathbf{Z}^{\leq -1}}\}$ , and analogously for  $Q_{\bar{x}}$ .

For the explicit formula, unravel our construction. For  $x$  and  $\bar{x}$  we may have picked different constants  $K$ , but in view of Assumption  $\mathfrak{C}^{\pm}$ , taking the minimum of these choices, will be fine for both.  $\square$

## 5. EQUIDISTRIBUTION ARGUMENTS

Let us recall Weyl Equidistribution for  $n$  variables.

**Theorem 5.1** (Weyl Equidistribution). *Suppose  $(t_n)_{n \geq 1}$  is a uniformly distributed sequence in  $[0, 1]^d$ . Then for every Riemann-integrable  $f : [0, 1]^d \rightarrow \mathbf{R}$ , one has equality*

$$(5.1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(t_n) = \int_{[0, 1]^d} f(\underline{s}) \, d\underline{s}.$$

This type of equidistribution statement will be an important tool later. However, we will need to apply it in situations where the function  $f$  is not Riemann-integrable.

*Remark 5.2.* This is a considerable problem even for  $d = 1$ : The left-hand side only depends on countably many values of  $f$ , so any notion of integrability which is preserved under changing  $f$  at countably many points is inevitably too weak to keep the conclusion of the theorem intact. There is a clarifying *No-Go Theorem*: Given any Lebesgue-integrable function  $f : [0, 1] \rightarrow \mathbf{R}$  which does *not* admit a Riemann-integrable representative, there must exist a uniformly distributed sequence  $(t_n)$  such that Equation 5.1 fails [dBP68].

There are more refined and flexible versions of equidistribution theorems which allow relaxing the assumption of Riemann-integrability when working with  $d = 1$  and the sequence is

$$t_n := \{n\theta\},$$

where we write

$$\{x\} := x - \lfloor x \rfloor$$

for the fractional part of a real number  $x$ .

*Remark 5.3.* If  $\theta$  is irrational, this sequence stems from an ergodic discrete dynamical system on the unit circle, so one can get a result similar to Equation 5.1 by Birkhoff's Ergodic Theorem, however it is only valid almost everywhere, and that is not good enough for our purposes.

Inspired by work of Hardy and Littlewood, Oskolkov introduced his notion of functions of Class H and identified conditions for Equation 5.1 to hold, [Osk90], [Osk94]. We shall work with a stronger version due to Baxa and Schoißengeier. We will also have to do a little extra work since we need the result to hold for the sequence  $(\{n\theta_1\}, \{n\theta_2\})$ , i.e. dimension  $d = 2$ , as well. Either way, the specific arithmetic properties of the number  $\theta$  become relevant, so we need to recall some material:

Suppose  $\theta \in (0, 1)$  is a real number. The (simple) continued fraction presentation of  $\theta$  is

$$(5.2) \quad \theta = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}, \quad (\text{with } a_n \in \mathbf{Z}_{\geq 1})$$

and is customarily abbreviated by  $\theta = [a_1, a_2, \dots]$ . The  $a_n$  are the *partial quotients*. Moreover, one defines  $p_n, q_n$  recursively by

$$\begin{aligned} p_0 &:= 0, & q_0 &:= 1, & p_{-1} &:= 1, & q_{-1} &:= 0, \\ p_{n+1} &:= a_{n+1}p_n + p_{n-1}, & q_{n+1} &:= a_{n+1}q_n + q_{n-1} \end{aligned}$$

for all  $n \in \mathbf{Z}_{\geq 0}$ . The fractions  $\frac{p_n}{q_n}$  are the *convergents* and they correspond to truncating the continued fraction in line 5.2 after evaluating the  $n$ -th interwoven fraction. They are always in lowest terms, i.e.  $(p_n, q_n) = 1$ .

The right-hand side in line 5.2 thus has the tacit meaning to be the limit of the sequence of convergents (one can show that this always converges). If  $\theta \in (0, 1)$  is irrational, its simple continued fraction always exists and the  $a_1, a_2, \dots$  are uniquely determined [Khi97, Chapter B], and conversely all sequences  $(a_n)$  in  $\mathbf{Z}_{\geq 1}$  define an irrational  $\theta \in (0, 1)$ .

**Definition 5.4.** For every real number  $\theta$ , let  $\|\theta\| \in [0, \frac{1}{2}]$  denote the distance to the closest integer. An irrational  $\theta \in (0, 1)$  is called *badly approximable* if one (then all) of the following equivalent conditions are met:

- (1) The infimum  $\inf_{n \geq 1} \{n \|n\theta\|\} > 0$  is strictly positive.
- (2) There exists a  $C_\theta > 0$  such that  $\left| \theta - \frac{p}{q} \right| > \frac{C_\theta}{q^2}$  holds for all  $p/q \in \mathbf{Q}$ .
- (3) There exists a  $K_\theta > 0$  such that  $a_n < K_\theta$  holds for all partial quotients  $a_n$  in the continued fraction  $[a_1, a_2, \dots]$ .

The equivalence of these characterizations is shown in [Khi97, Theorem 23] or [Sch80, Theorem 5F]. We also need:

**Lemma 5.5.** Suppose  $1, \theta_1, \dots, \theta_d$  are  $\mathbf{Q}$ -linearly independent elements inside the real line. Then the sequence of vectors

$$(\{n\theta_1\}, \dots, \{n\theta_d\})_{n \geq 1} \in [0, 1]^d$$

is uniformly distributed.

*Proof.* Standard. A detailed proof is given for example in [KN74, Ch. I, Theorem 6.3, Example 6.1].  $\square$

We fix  $\theta \in (0, 1)$  irrational. We shall need to work with the Main Lemma of Baxa and Schoißengeier, so let us recall its statement:

**Definition 5.6** ([BS02]). *For the fixed irrational  $\theta \in (0, 1)$ , we use the following notation: Given any  $N \in \mathbf{Z}_{\geq 1}$ , let  $\sigma_N \in \mathfrak{S}_N$  denote the permutation such that*

$$\{\sigma_N(1)\theta\} < \{\sigma_N(2)\theta\} < \{\sigma_N(3)\theta\} < \cdots < \{\sigma_N(N)\theta\}.$$

With this notation:

**Lemma 5.7** (Baxa–Schoißengeier Main Lemma). *Fix  $\theta \in (0, 1)$  irrational and use the notation as introduced above. Let  $\beta = \frac{p}{q} \in \mathbf{Q} \cap (0, 1]$  be a rational number given in lowest terms (with  $p, q > 0$ ). Write  $n_N$  to denote the largest integer such that*

$$1 \leq n_N \leq N \quad \text{and} \quad \{\sigma_N(n_N)\theta\} < \beta.$$

*Then there exists  $m_0 \in \mathbf{Z}_{\geq 1}$  such that for all  $m \geq m_0$  and all integers  $b$  with  $1 \leq b \leq \max\{1, \frac{a_{m+1}}{2q}\}$  the following holds: For every  $f : [0, 1] \rightarrow [0, +\infty)$  a Lebesgue-integrable function, which is monotonously increasing in  $[0, \beta)$  and is zero in  $(\beta, 1]$*

$$\frac{1}{N} \sum_{n=1}^N f(\{n\theta\}) \leq 7q \int_0^1 f(t) dt + \frac{1}{\sigma_N(n_N)} f(\{\sigma_N(n_N)\theta\})$$

*holds for  $N := bq_m$ .*

This is [BS02, Main Lemma]. Next, we discuss the Equidistribution theorem of *loc. cit.* Following their paper, we isolate a well-behaved class of real functions that go considerable beyond the Riemann-integrable ones:

**Definition 5.8** ([BS02, Theorem 1]). *Let  $F \subseteq [0, 1] \cap \mathbf{Q}$  be a finite subset of the rational numbers. We say that a function  $f : [0, 1] \rightarrow \mathbf{R}$  belongs to Class BS( $F$ ) if the following properties hold:*

- (1)  *$f$  is Lebesgue-integrable,*
- (2)  *$f$  is almost everywhere continuous,*
- (3)  *$f$  locally bounded at every point in  $[0, 1] \setminus F$ ,*
- (4) *for every  $\beta \in F$ , there exists some  $\varepsilon > 0$  such that  $f|_{(\beta-\varepsilon, \beta)}$  is bounded or monotone,*
- (5) *for every  $\beta \in F$ , there exists some  $\varepsilon > 0$  such that  $f|_{(\beta, \beta+\varepsilon)}$  is bounded or monotone.*

*We call  $F$  the set of (possible) singularities. If  $f : [0, 1] \rightarrow \mathbf{C}$  is complex-valued, we say  $f \in \text{BS}(F)$  if both real and imaginary part belong to BS( $F$ ).*

**Theorem 5.9** (Baxa–Schoißengeier Equidistribution). *Suppose  $\theta \in (0, 1)$  is irrational and  $f \in \text{BS}(F)$  for a suitably chosen  $F$ . Then we have equality*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta\}) = \int_0^1 f(t) dt$$

*if and only if*

$$(5.3) \quad \lim_{n \rightarrow \infty} \frac{f(\{n\theta\})}{n} = 0.$$

*The condition of Equation 5.3 is automatically satisfied if  $\theta$  is badly approximable.*

Most of this is [BS02, Theorem 1]. The last claim is [BS02, Theorem 2, (2)]. An even stronger version was established in [Bax05], under broader, but very technical assumptions.

*Remark 5.10.* If one can ensure stronger conditions on  $f$ , a precursor of this result is given in [Osk90]. If  $f$  is a function of Oskolkov's Class H and  $\theta$  is badly approximable, its partial quotients are bounded, so  $\{n\theta\}$  is regularly distributed by [Osk94, Theorem 3]. Then Theorem 1 *loc. cit.* also gives the same conclusion.

**5.1. An ad-hoc multi-dimensional equidistribution theorem.** As mentioned before, we shall need a slightly stronger form of the Baxa–Schoißengeier result, imitating Weyl Equidistribution not just for  $d = 1$ , but also for  $d = 2$ .

**Definition 5.11.** Let  $F \subseteq [0, 1] \cap \mathbf{Q}$  be a finite subset of the rational numbers. We say that a function  $f : [0, 1]^d \rightarrow \mathbf{R}$  belongs to Class  $\text{BSU}^d(F)$  if the following holds: There exists a function  $g \in \text{BS}(F)$  and a function  $h : [0, 1]^d \rightarrow \mathbf{R}$  such that

$$f(y_1, \dots, y_d) = g(y_1) \cdot h(y_1, \dots, y_d),$$

where the function  $h$  is Riemann-integrable. We call a choice of such a function  $g$  a singular weight. If  $f : [0, 1]^d \rightarrow \mathbf{C}$  is complex-valued, we say  $f \in \text{BSU}^d(F)$  if both real and imaginary part belong to  $\text{BSU}^d(F)$ .

We are ready to state our minimalistic extension of Baxa–Schoißengeier Equidistribution, just about strong enough for what we need:

**Theorem 5.12** (Ad-hoc Unidirectional Equidistribution). Suppose  $1, \theta_1, \dots, \theta_d$  are  $\mathbf{Q}$ -linearly independent real numbers. Suppose  $F \subseteq [0, 1] \cap \mathbf{Q}$  is finite. Suppose  $f : [0, 1]^d \rightarrow \mathbf{R}$  is a function in Class  $\text{BSU}^d(F)$  which admits a singular weight  $g$  such that

$$(5.4) \quad \lim_{n \rightarrow \infty} \frac{g(\{n\theta_1\})}{n} = 0.$$

Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta_1\}, \dots, \{n\theta_d\}) = \int_{[0, 1]^d} f(\underline{s}) \, d\underline{s}.$$

We follow the proof of the one-dimensional case in [BS02] very closely, but need to perform some minor modifications. We have tailored this formulation to be sufficient for our purposes. It would be desirable to have more general theorems of this sort.

*Proof.* We write  $f(y_1, \dots, y_d) = g(y_1) \cdot h(y_1, \dots, y_d)$  with  $g$  the singular weight such that Equation 5.4 holds. We may write  $h = h^+ - h^-$  for  $h^\pm$  non-negative functions. Since left- and right-hand side of our claim are linear, it suffices to prove our claim under the additional assumption  $h \geq 0$ . As  $h$  is Riemann-integrable by assumption, thus bounded, we get

$$0 \leq h(y_1, \dots, y_d) \leq h_{\max}.$$

Without loss of generality, we may assume  $\theta_1 \in (0, 1)$ . By assumption,  $\theta_1$  is linearly independent from 1 over the rationals, so  $\theta_1$  is irrational. Henceforth, we use the notation  $p_i, q_i, a_i$  etc. for the convergents, partial quotients, etc. for  $\theta := \theta_1$ .

(Step 1) As in [BS02], we first deal with the case  $F = \emptyset$ . By Lebesgue's integrability criterion, a bounded function  $f : [0, 1]^d \rightarrow \mathbf{R}$  is Riemann-integrable if and only if it is continuous almost everywhere. Thus, as  $F = \emptyset$ ,  $f$  is locally bounded on the compactum  $[0, 1]^d$  and thus bounded. By Lebesgue's criterion, it follows that  $f$  is Riemann-integrable and we can use multi-dimensional Weyl Equidistribution, Theorem 5.1, since by Lemma 5.5

$(\{n\theta_1\}, \dots, \{n\theta_d\})$  is uniformly distributed in  $[0, 1]^d$ .

(Step 2) Next, as in [BS02], consider the case  $F = \{\beta\}$ . Assume that  $\beta = \frac{p}{q} \in (0, 1]$  is a rational number in lowest terms,  $p, q > 0$ . Suppose  $\lim_{t \rightarrow \beta, t < \beta} g(t) = +\infty$  and  $g|_{(\beta, 1)} = 0$ . Define an *admissible*  $\varepsilon > 0$  to be any element in

$$\{\varepsilon \mid \varepsilon > 0, \text{ the function } g|_{[\beta-\varepsilon, \beta)} \text{ is monotone and non-negative}\}.$$

By the assumptions of BS( $F$ ), admissible  $\varepsilon$  exist. For any sufficiently large integer  $N$ , we may choose some  $m \in \mathbf{Z}_{\geq 1}$  with  $q_m \leq N < q_{m+1}$ , and then pick  $b := \lfloor N/q_m \rfloor \geq 1$  (cf. renewal time). For every admissible  $\varepsilon$ , define

$$f_\varepsilon(y_1, \dots, y_n) := f(y_1, \dots, y_n) \cdot c_{[\beta-\varepsilon, 1] \times [0, 1]^{d-1}} \quad g_\varepsilon(y_1) := g(y_1) \cdot c_{[\beta-\varepsilon, 1]},$$

where  $c_I$  denotes the characteristic function of a set  $I$ . Then

$$f_\varepsilon(y_1, \dots, y_n) = g_\varepsilon(y_1)h(y_1, \dots, y_n).$$

Moreover, since  $g_\varepsilon$  and  $h$  are non-negative (by the choice of  $\varepsilon$ ),  $f_\varepsilon$  is non-negative. Thus,

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N f_\varepsilon(\{n\theta_1\}, \dots, \{n\theta_d\}) &= \frac{1}{N} \sum_{n=1}^N g_\varepsilon(\{n\theta_1\}) \cdot h(\{n\theta_1\}, \dots, \{n\theta_d\}) \\ &\leq \frac{h_{\max}}{N} \sum_{n=1}^N g_\varepsilon(\{n\theta_1\}), \end{aligned}$$

Now Baxa and Schoißengeier perform a case distinction, using their Main Lemma. We copy this: If  $\frac{a_{m+1}}{2q} - 1 < b$ , the Main Lemma yields:

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N g_\varepsilon(\{n\theta_1\}) &\leq \quad ( \dots ) \\ &\leq 28q^2 \int_0^1 g_\varepsilon(s) ds + \frac{4q}{\sigma_{q_{m+1}}(n_{q_{m+1}})} g_\varepsilon(\{\sigma_{q_{m+1}}(n_{q_{m+1}})\theta_1\}), \end{aligned}$$

with  $\sigma_{(-)}, n_{(-)}$  as in the sense of the Main Lemma. Or, in the other case  $b \leq \frac{a_{m+1}}{2q} - 1$ , it yields

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N g_\varepsilon(\{n\theta_1\}) &\leq \quad ( \dots ) \\ &\leq 14q \int_0^1 g_\varepsilon(s) ds + \frac{2}{\sigma_{(b+1)q_m}(n_{(b+1)q_m})} g_\varepsilon(\{\sigma_{(b+1)q_m}(n_{(b+1)q_m})\theta_1\}). \end{aligned}$$

We refer to their paper for any details. As we can do this for a sequence of choices  $N$  with  $N \rightarrow +\infty$ , and we have  $\lim_{N \rightarrow +\infty} \sigma_N(n_N) = +\infty$  (cf. Definition 5.6), we may use Equation 5.4 and as in [BS02] we obtain: For every admissible  $\varepsilon$ , we have the upper bound

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N f_\varepsilon(\{n\theta_1\}, \dots, \{n\theta_d\}) &\leq \limsup_{N \rightarrow +\infty} \frac{h_{\max}}{N} \sum_{n=1}^N g_\varepsilon(\{n\theta_1\}) \\ &\leq 28 \cdot h_{\max} \cdot q^2 \int_0^1 g_\varepsilon(s) ds. \end{aligned}$$

Now, since we had assumed that  $F = \{\beta\}$ , the function  $g - g_\varepsilon$  is Riemann-integrable, thus

$$f - f_\varepsilon = (g - g_\varepsilon) \cdot h$$

is itself Riemann-integrable. Multi-dimensional Weyl Equidistribution applies so that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N (f - f_\varepsilon)(\{n\theta_1\}, \dots, \{n\theta_d\}) = \int_{[0,1]^d} (f - f_\varepsilon)(\underline{s}) d\underline{s} \leq \int_{[0,1]^d} f(\underline{s}) d\underline{s}$$

since  $f_\varepsilon$  is a non-negative function, and  $f$  is Lebesgue-integrable. Combining both upper bounds, we obtain

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta_1\}, \dots, \{n\theta_d\}) \leq \int_{[0,1]^d} f(\underline{s}) d\underline{s} + 28 \cdot h_{\max} \cdot q^2 \cdot \int_0^1 g_\varepsilon(s) ds$$

for all admissible  $\varepsilon$ . Thus,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta_1\}, \dots, \{n\theta_d\}) \leq \int_{[0,1]^d} f(\underline{s}) d\underline{s}.$$

Conversely, since  $f_\varepsilon$  is non-negative,

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N f(\{n\theta_1\}, \dots, \{n\theta_d\}) \geq \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N (f - f_\varepsilon)(\{n\theta_1\}, \dots, \{n\theta_d\}) = \int_{[0,1]^d} f(\underline{s}) d\underline{s}$$

by Weyl Equidistribution and since  $f - f_\varepsilon$  is Riemann-integrable. As both limes superior and inferior exist and coincide, we obtain that the limit exists and is of said value.

(Step 3) Now one can do an induction over the cardinality of  $F$ . This argument can be carried out precisely as in [BS02] and we leave it to the reader.  $\square$

## 6. THE ORTHOGONALITY THEOREM

This section is devoted to the proof of the following statement.

**Theorem 6.1** (Orthogonality). *Suppose  $\theta$  is a real number such that either*

- $e^{2\pi i \theta}$  is an algebraic number, or
- $\theta$  is badly approximable, i.e. it has a bounded sequence of partial quotients.

*Then the following holds:*

- (1) *If  $\dim_{\mathbf{Q}} \langle 1, \theta \rangle = 2$ : For all  $m \in \mathbf{Z}$ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| \cdot e^{2\pi i m n \theta} = -\frac{1}{2|m|} \delta_{m \neq 0}.$$

*If  $m \in \mathbf{Q} \setminus \mathbf{Z}$ , we get a value*

$$C_m \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

*where  $\chi$  ranges over a set of non-principal Dirichlet characters modulo  $2v$  for  $v \geq 1$  the denominator of  $m$  in lowest terms. The values  $C_m$  only depend on  $m$ , and are independent of  $\theta$ .*

- (2) *If  $\alpha$  is a real number and  $\dim_{\mathbf{Q}} \langle 1, \theta, \alpha \rangle = 3$ , then for all  $m \in \mathbf{Z}$ ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| \cdot e^{2\pi i m n \alpha} = 0.$$

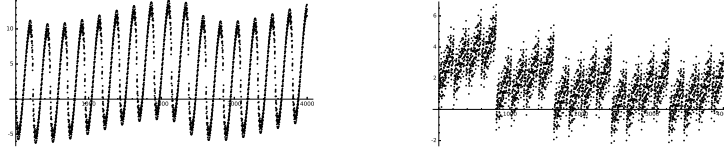
**Example 6.2.** *Let us look at the following polynomial*

$$f := x^8 - 120x^7 + 4332x^6 - 86664x^5 + 1311590x^4 - 10994952x^3 + 75494124x^2 - 19704x + 1.$$

*It was constructed by A. Dubickas [Dub14]. It has a special property: Four complex roots of this polynomial have the same absolute value (this is not at all obvious) and are (obviously) algebraic units. Thus, the quotients  $\frac{v_1}{v_2}$  of any two such are concrete complex algebraic numbers lying on the unit circle. Moreover,  $\frac{v_1}{v_2}$  will not be a root of unity, so Claim (1) of the theorem applies. We get<sup>3</sup>*

$$\begin{aligned} u_1 &:= e^{2\pi i \theta_1} & \text{with} & & \theta_1 &\approx 0.400842 \\ u_2 &:= e^{2\pi i \theta_2} & \text{with} & & \theta_2 &\approx 0.410383. \end{aligned}$$

*The sequence  $\sum_{n=1}^N \log |1 - e^{2\pi i n \theta_i}|$  (for  $i = 1, 2$ ) oscillates rather wildly, so for example for  $m = 0$ , the theorem claims that the average values of the sequence still balance out around zero. Even looking at concrete values, this is not obvious:*



The main idea will be to use equidistribution results to translate the statement of the Orthogonality Theorem into statements about integrals. So first of all, let us compute the relevant integrals.

**6.1. Integral values.** Define

$$P_m := \int_0^\pi \log(\sin t) e^{2imt} dt \quad \text{for} \quad m \in \mathbf{Z}.$$

**Lemma 6.3.** *We have  $P_0 = -\pi \log 2$  and  $P_m = -\frac{1}{2} \frac{\pi}{|m|}$  for all  $m \in \mathbf{Z} \setminus \{0\}$ .*

*Proof.* For  $m = 0$ , this is a not completely trivial, but still standard exercise in contour integration, presented for example in [Ahl78, Ch. 4, §5.3.5]. (Step 1) For non-negative  $m \geq 0$ , the same technique works with the appropriate changes made. We give the details for the sake of completeness: Fix some  $m \in \mathbf{Z}_{\geq 0}$ . Consider the function

$$g(z) := \log(1 - e^{2iz}) e^{2imz}.$$

As in [Ahl78, Ch. 4, §5.3.5], it is easy to see that  $1 - e^{2iz} \in \mathbf{R}_{\leq 0}$  holds if and only if  $z \in \pi\mathbf{Z} + i\mathbf{R}_{\leq 0}$ . Pick some  $K > 0$  and consider the box  $\text{Box} := [0, \pi] \times i[0, K]$ . Starting from the contour  $\partial\text{Box}$ , in order to obtain that  $g$  is holomorphic everywhere inside and on a neighbourhood of the contour, we need to switch to a modified contour  $C$ : Introduce small quarter-circle indentations at  $z = 0$  and  $z = \pi$ . Then by Cauchy's Integral Theorem,

$$0 = \int_C g(z) dz = \int_{C_{\text{top}}} g(z) dz + \int_0^\pi \log(1 - e^{2it}) e^{2imt} dt,$$

where  $C_{\text{top}}$  is the top edge. The contributions from the left and right edge cancel out by the periodicity  $g(z + \pi) = g(z)$ . Under  $K \rightarrow +\infty$ , the top edge integral converges to

<sup>3</sup>as an aside: note that by Gelfond–Schneider, both  $\theta_1, \theta_2$  are necessarily transcendental.

zero (this needs  $m \geq 0$ ). The treatment of the quarter-circle indentations requires limit considerations, we refer to loc. cit. for the details. Finally, Euler's formula immediately gives  $1 - e^{2it} = -2ie^{iz} \sin z$  and thus

$$\begin{aligned} 0 &= \int_0^\pi \log(1 - e^{2it}) e^{2imt} dt = \int_0^\pi \log(-2ie^{it} \sin t) e^{2imt} dt \\ &= \log(-2i) \int_0^\pi e^{2imt} dt + \int_0^\pi \log(e^{it}) e^{2imt} dt + \int_0^\pi \log(\sin t) e^{2imt} dt. \end{aligned}$$

The second line follows from the first by the functional equation for the complex logarithm (it requires a little side thought to be sure that no branch switch discrepancy of  $2\pi i$  gets introduced this way). We have  $\log(-2i) = \log 2 - \frac{\pi i}{2}$  and  $\int_0^\pi e^{2imt} dt = \delta_{m=0} \pi$ . Moreover,  $\log(e^{it}) = it$  as  $t \in (0, \pi)$ . Hence,

$$\int_0^\pi \log(e^{it}) e^{2imt} dt = i \int_0^\pi t e^{2imt} dt = \begin{cases} \frac{1}{2} \frac{\pi}{m} & \text{for } m \neq 0 \\ \frac{1}{2} i \pi^2 & \text{for } m = 0 \end{cases}$$

by straight-forward partial integration. Combining these computations immediately gives our claim for all  $m \geq 0$ . (*Step 2*) Next, we deal with  $m \in \mathbf{Z}_{\geq -1}$ . The contour integration in *Step 1* does not work anymore (Reason: As one moves the top contour off to infinity, this contribution does no longer converge to zero. To see this, note that  $|g(iy)| = |\log(1 - e^{-2y}) e^{-2my}|$ , and for negative  $m$  the term  $|e^{-2my}|$  grows exponentially as  $y \rightarrow +\infty$ ). We resolve this as follows: We have

$$P_{-m} = \int_0^\pi \log(\sin t) \overline{e^{2imt}} dt = \overline{P_m}$$

for all  $m$ . Since *Step 1* shows that for  $m \geq 1$  the value of  $P_m$  is real, it is not affected by complex conjugation.  $\square$

Next, define auxiliary values

$$(6.1) \quad W_m := \int_0^1 \log |1 - e^{2\pi i \theta}| e^{2\pi i m \theta} d\theta \quad \text{for } m \in \mathbf{Z}.$$

Recall the standard sine squaring formula,  $\sin(t)^2 = \frac{1}{2} (1 - \cos(2t))$ . It implies  $2 \sin\left(\frac{t}{2}\right)^2 = (1 - \cos t)$  and thus

$$|1 - e^{it}|^2 = (1 - e^{it}) \overline{(1 - e^{it})} = 2 - 2 \cos t = 4 \sin\left(\frac{t}{2}\right)^2.$$

Then,

$$(6.2) \quad \log |1 - e^{it}| = \frac{1}{2} \log \left( 4 \sin\left(\frac{t}{2}\right)^2 \right) = \log \left( 2 \sin\left(\frac{t}{2}\right) \right).$$

**Lemma 6.4.** *We have  $W_m = -\frac{1}{2|m|} \delta_{m \neq 0}$  for all  $m \in \mathbf{Z}$ .*



*Proof.* By substitution, we switch from the variable  $\theta$  to  $\theta/2\pi$ . Then use Formula 6.2 in order to obtain

$$\begin{aligned} W_m &= \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| e^{im\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \left( 2 \sin \left( \frac{\theta}{2} \right) \right) e^{im\theta} d\theta \\ &= \frac{1}{\pi} \int_0^\pi \log (2 \sin \theta) e^{2im\theta} d\theta \\ &= \frac{1}{\pi} \int_0^\pi \log(\sin \theta) \cdot e^{2im\theta} d\theta + \frac{1}{\pi} \log 2 \int_0^\pi e^{2im\theta} d\theta \\ &= \frac{1}{\pi} P_m + \frac{1}{\pi} \log 2 \cdot \pi \delta_{m=0} = -\frac{1}{2|m|} \delta_{m \neq 0}. \end{aligned}$$

Here we have used the functional equation of the real logarithm, which yields a term of the shape  $P_m$  and then we may evaluate the entire expression using Lemma 6.3.  $\square$

**6.2. Fractional values.** So far, we have determined  $W_m$  for all integer values. For non-integral values, the structure is more complicated. We will analyze this case now. Let  $m \in \mathbf{R} \setminus \mathbf{Z}_{\leq -1}$  be given. Recall that  $w^m = \exp(m \cdot \log w)$  as a function in  $w$  is holomorphic on  $\mathbf{C} \setminus (-\infty, 0]$  and  $\log(1 - w)$  is holomorphic in  $\mathbf{C} \setminus [1, \infty)$ . The intersection  $X := \mathbf{C} \setminus ((-\infty, 0] \cup [1, \infty))$  is simply connected, so

$$S_m(z) := - \int_0^z \log(1 - w) w^m \frac{dw}{w}$$

determines a well-defined holomorphic function on  $X$ , independent of the choice of a path of integration from 0 to  $z$ . For  $m = 0$ , we have  $S_0(z) = \text{Li}_2(z)$ , the classical dilogarithm. For  $|z| < 1$ , termwise integration of the logarithm series yields the uniformly convergent series

$$S_m(z) = \sum_{r=1}^{\infty} \frac{1}{r} \frac{z^{r+m}}{r+m} \quad \text{for} \quad z \in X, |z| < 1.$$

Note that this will usually *not* be a power series since  $m$  need not be a natural number. We may use this series to attach a value to the two points  $\{-1, 1\} \notin X$ , namely

$$(6.3) \quad S_m(1) := \sum_{r=1}^{\infty} \frac{1}{r(r+m)} \quad \text{and} \quad S_m(-1) := \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{i\pi(r+m)}}{r+m}.$$

Note that these values really hinge on our choice of preferred branches, e.g.

$$(6.4) \quad S_m(-1) = \lim_{\substack{z \rightarrow -1, \\ z \in X, \text{Im } z > 0}} S_m(z).$$

**Proposition 6.5.** *Suppose  $m \in \mathbf{R} \setminus \mathbf{Z}_{\leq 0}$ .*

(1) *Then*

$$i \int_0^\pi \log \left( 2 \sin \left( \frac{\theta}{2} \right) \right) \cdot e^{im\theta} d\theta = S_m(1) - S_m(-1) - \frac{1}{2m^2} (1 - e^{i\pi m} + m\pi i),$$

where  $S_m(1)$  and  $S_m(-1)$  are defined as in Equation 6.3.

(2) *Moreover,*

$$i \int_\pi^{2\pi} \log \left( 2 \sin \left( \frac{\theta}{2} \right) \right) \cdot e^{im\theta} d\theta = -e^{2\pi im} \cdot i \int_0^\pi \log \left( 2 \sin \left( \frac{\theta}{2} \right) \right) \cdot e^{im\theta} d\theta,$$

reducing this integral to (1).



*Proof.* Define  $z := e^{i\pi \frac{j}{n}}$  (we shall only need the case  $n = j = 1$ , but dealing with the general case makes the computation clearer). Let  $C \in \{0, m\}$  be arbitrary. For all  $s \in \mathbf{C}$  with  $\operatorname{Re} s > 1$ , we compute

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{z^r}{(r+C)^s} &= \sum_{r=0}^{\infty} \sum_{l=1}^{2n} \frac{z^{2rn+l}}{(2rn+l+C)^s} = \sum_{l=1}^{2n} (e^{i\pi \frac{j}{n}})^l \sum_{r=0}^{\infty} \frac{1}{(2rn+l+C)^s} \\ &= \frac{1}{2n} \sum_{l=1}^{2n} (e^{i\pi \frac{j}{n}})^l \sum_{r=0}^{\infty} \frac{1}{(r + \frac{l+C}{2n})^s} = \frac{1}{2n} \sum_{l=1}^{2n} (e^{i\pi \frac{j}{n}})^l \zeta\left(s, \frac{l+C}{2n}\right), \end{aligned}$$

where  $\zeta(s, A) := \sum_{r=0}^{\infty} (r+A)^{-s}$  denotes the Hurwitz zeta function. Note that we have used that  $m \notin \mathbf{Z}_{\leq -1}$ . It is well-known that the Hurwitz zeta function at *rational* parameters, can be expressed through Dirichlet  $L$ -values. Concretely,

$$\zeta\left(s, \frac{a}{b}\right) = \frac{b^s}{\varphi(b)} \sum_{\chi} \overline{\chi(a)} \cdot L(s, \chi),$$

where  $\chi$  runs through all Dirichlet characters modulo  $b$ , and  $\varphi$  is Euler's totient function. Thus, we may expand

$$\sum_{r=1}^{\infty} \frac{z^r}{(r+C)^s} = \sum_{i \in I} x_i \cdot L(s, \chi^{(i)}),$$

for  $I$  some finite index set,  $x_i \in \overline{\mathbf{Q}}$ ,  $\chi^{(i)}$  Dirichlet characters modulo  $2n$  (for  $C = 0$ ) resp.  $2nv$  (for  $C = \frac{u}{v}$ ). Now, restrict to the case  $n = j = 1$ . Since  $|z| = 1$ , but  $z \neq 1$ , the limit of the left-hand side for  $s \rightarrow 1$  exists. For all non-principal characters  $\tilde{\chi}$ ,  $L(s, \tilde{\chi})$  exists for  $s = 1$  on the right-hand side. Thus, the principal character  $\chi_0$  does not occur among those  $i \in I$  with  $x_i \neq 0$  (Reason: Suppose it does. Since all other summands have a finite limit for  $s \rightarrow 1$ , this would force  $L(s, \chi_0)$  to have a finite limit for  $s \rightarrow 1$  as well, but there is a pole instead). Thus, we can actually carry out the limit  $s \rightarrow 1$  and obtain

$$(6.6) \quad \sum_{r=1}^{\infty} \frac{(e^{i\pi})^r}{r+C} = \sum_{i \in I} x_i \cdot L(1, \chi^{(i)})$$

for a collection of non-principal Dirichlet characters  $\chi^{(i)}$  modulo 2 (for  $C = 0$ ) resp.  $2v$  (for  $C = \frac{u}{v}$ ). (Of course, there is only one such Dirichlet character for  $C = 0$ , but let us ignore this simplification). Finally, note that we have

$$\frac{1}{r(r+m)} = \frac{1}{m} \left( \frac{1}{r} - \frac{1}{r+m} \right)$$

as  $m \neq 0$ . Thus, for  $|w| < 1$  we have

$$S_m(w) = \frac{w^m}{m} \left( \sum_{r=1}^{\infty} \frac{w^r}{r} - \sum_{r=1}^{\infty} \frac{w^r}{r+m} \right).$$

We get  $S_m(-1)$  if we plug in  $w := e^{i\pi}$ . Although this does not satisfy  $|w| < 1$ , it is consistent with our definition of  $S_m(-1)$  by line 6.4 and the series are conditionally convergent. Since  $w^m \in \overline{\mathbf{Q}}$ , Equation 6.6 implies our claim.  $\square$

If  $\Gamma$  denotes the Gamma function, the *digamma function* is defined as its logarithmic derivative, i.e.

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

The standard functional equation of the Gamma function implies that

$$(6.7) \quad \psi(z+1) = \psi(z) + \frac{1}{z}.$$

Write  $\gamma$  for the Euler–Mascheroni constant. The Weierstrass product formula for the Gamma function yields

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{r=1}^{\infty} \left(1 + \frac{z}{r}\right) e^{-\frac{z}{r}}.$$

Thus, taking its logarithmic derivative,

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{r=1}^{\infty} \left( \frac{1}{r+z} - \frac{1}{r} \right) = \gamma + \frac{1}{z} - z \sum_{r=1}^{\infty} \frac{1}{(r+z)r}.$$

Hence, by the functional equation of Equation 6.7,

$$(6.8) \quad \frac{(\psi(z) + \gamma) + \frac{1}{z}}{z} = \sum_{r=1}^{\infty} \frac{1}{(r+z)r}.$$

**Proposition 6.7.** *For a fraction  $m = \frac{u}{v} \in \mathbf{Q} \setminus \mathbf{Z}$  (with  $u, v \in \mathbf{Z}$  and  $v \geq 1$ ), the value of  $W_m$  lies in the field*

$$\mathbf{Q}(\mu_{\infty}, \pi, \{L(1, \chi)\}_{\chi}),$$

where  $\chi$  runs through a finite set of non-principal Dirichlet characters modulo  $2v$ .

*Proof.* We have

$$W_m = \frac{1}{2\pi} \left( \int_0^{\pi} + \int_{\pi}^{2\pi} \right) \log \left( 2 \sin \left( \frac{\theta}{2} \right) \right) e^{im\theta} d\theta,$$

so  $W_m$  is a sum of two terms whose shape we understand thanks to Prop. 6.5. In this presentation, we just have algebraic numbers in a cyclotomic field,  $\pi$ ,  $S_m(-1)$  whose structure is settled by Lemma 6.6, and  $S_m(1)$ . Finally, by Equation 6.8,

$$S_m(1) = \sum_{r=1}^{\infty} \frac{1}{r(r+m)} = \frac{(\psi(m) + \gamma) + \frac{1}{m}}{m}.$$

Now,  $\psi(m) + \gamma$  for  $m = \frac{u}{v} \in \mathbf{Q} \setminus \mathbf{Z}$  can be written as a  $\mathbf{Q}(\mu_{\infty})$ -linear combination of values  $\log(1 - \zeta_v^i)$  for  $\zeta_v$  a primitive  $v$ -th root of unity [MS07, Lemma 21], and these in turn as  $L(1, \chi)$  for suitable  $\chi$ .  $\square$

*Aside 6.8.* Suppose  $m \in \mathbf{Q} \setminus \mathbf{Z}$ . Then it was proven by Bundschuh that  $\psi(m) + \gamma$  is transcendental. This is [Bun79, Korollar 1]. More is known, e.g. on the linear independence of such values over  $\mathbf{Q}$ , [MS07, Theorem 4] (however, over  $\mathbf{Q}(\mu_{\infty})$  the situation is less clear).

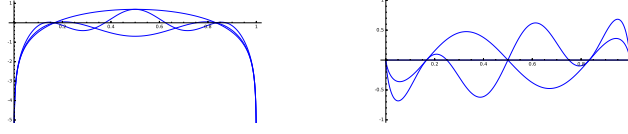
### 6.3. Proof of Theorem 6.1.

*Proof of Theorem 6.1. (Claim 1)* We claim that the function

$$(6.9) \quad f(t) := \log |1 - e^{2\pi it}| \cdot e^{2\pi imt} \quad \text{for } t \in (0, 1)$$

satisfies  $f \in \text{BS}(\{0, 1\})$ : It is clearly continuous on  $(0, 1)$ . Next, treat the real and imaginary parts separately. They have a very different boundary behaviour, as witnessed by the

following figure of the graphs for  $m = 0, 1, 2$ .



The real part is  $\operatorname{Re} f(t) = \log |1 - e^{2\pi i t}| \cdot \cos(2\pi m t)$ . For the limit  $t \rightarrow 0$ , we easily find that we have  $\log |1 - e^{2\pi i t}| \rightarrow -\infty$  and since  $\cos(0) = +1$  and  $r > 0$ , we get  $\lim_{t \rightarrow 0, t \in (0,1)} \operatorname{Re} f(t) = -\infty$ . For the limit  $t \rightarrow 1$ , the same happens. For all  $t \in (0, \frac{1}{2})$ , the function  $-\log |1 - e^{2\pi i t}|$  is monotonously decreasing, and for  $t > 0$  staying sufficiently small,  $\cos(2\pi m t)$  is also monotonously decreasing (or if  $m = 0$  constant). Either way, their product is monotonously decreasing. Thus, the negative is monotonously increasing. For  $t \in (\frac{1}{2}, 1)$ , proceed symmetrically. This finishes the real part. The imaginary part is  $\operatorname{Im} f(t) = \log |1 - e^{2\pi i t}| \sin(2\pi m t)$ . We compute, using Formula 6.2,

$$(6.10) \quad \lim_{t \rightarrow 0, t \in (0,1)} \operatorname{Im} f(t) = \lim_{t \rightarrow 0, t \in (0,1)} \log(2 \sin \pi t) \sin(2\pi m t).$$

We need a case distinction: If  $m = 0$ , this is constantly zero. In particular, the limit is zero. If  $m \neq 0$  so that  $\sin(2\pi m t) \neq 0$  near  $t = 0$ , we may rewrite this as

$$\lim_{t \rightarrow 0, t \in (0,1)} \log(2 \sin \pi t) \sin(2\pi m t) = \lim_{t \rightarrow 0, t \in (0,1)} \frac{\log(2 \sin \pi t)}{\left(\frac{1}{\sin(2\pi m t)}\right)},$$

and for  $t \rightarrow 0, t > 0$ , both numerator and denominator diverge to  $+\infty$ , so by the L'Hôpital Rule,

$$= - \lim_{t \rightarrow 0, t \in (0,1)} \frac{\cos(\pi t)}{2m \cos(2\pi m t)} \frac{\sin(2\pi m t)^2}{\sin(\pi t)}.$$

The first factor converges to some non-zero value, and since the sine is  $\sin(z) = z + O(z^2)$  to first order, the second factor tends to zero as  $t \rightarrow 0$ . It follows that the limit in line 6.10 is always zero. A very similar computation can be made for the limit  $t \rightarrow 1$ . We leave this to the reader. It follows that  $\operatorname{Im} f(t)$  may be continued to a continuous function on all of  $[0, 1]$ , so it even lies in  $\operatorname{BS}(\emptyset)$ .

This being settled, we observe that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| e^{2\pi i m n \theta} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i \{n\theta\}}| e^{2\pi i m \{n\theta\}}$$

and we want to apply Baxa–Schoißengeier Equidistribution, Theorem 5.9. To be able to invoke this result, it remains to check the condition

$$(6.11) \quad \lim_{n \rightarrow \infty} \frac{f(\{n\theta\})}{n} = 0.$$

(Case A) If  $u := e^{2\pi i \theta}$  is algebraic, it cannot be a root of unity, since that would contradict our assumption  $\dim_{\mathbf{Q}} \langle 1, \theta \rangle = 2$ . We also have  $|e^{2\pi i \theta}| = 1$ , so we can use the typical diophantine estimate: Namely,

$$\frac{|f(\{n\theta\})|}{n} = \frac{|\log |1 - e^{2\pi i n \theta}| e^{2\pi i m \{n\theta\}}|}{n} \leq \frac{1}{n} \log |1 - e^{2\pi i n \theta}| = \frac{1}{n} \log |1 - u^n|.$$

We have  $|u| = 1$  and since  $\theta$  is irrational,  $u$  cannot be a root of unity. Thus, Lemma 2.3 (which in turn hinges on the Gelfond estimate) is available and implies

$$\frac{|f(\{n\theta\})|}{n} \leq \frac{C_u \log n}{n}$$

for a suitable constant  $C_u > 0$  and this converges to zero for  $n \rightarrow +\infty$ . Thus, Equation 6.11 is fine.

(Case B) If  $\theta$  is badly approximable, Equation 6.11 automatically holds, see Theorem 5.9.

Thus, in either case, Theorem 5.9 applies. We get

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i \{n\theta\}}| e^{2\pi i m \{n\theta\}} = \int_0^1 \log |1 - e^{2\pi i t}| e^{2\pi i m t} dt = W_m$$

(with  $W_m$  as in line 6.1) and thus Lemma 6.4 and Prop. 6.7 yield our claim.

(Claim 2) The proof is similar, but now we work on  $[0, 1]^2$ . We have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| e^{2\pi i m n \alpha} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i \{n\theta\}}| e^{2\pi i m \{n\alpha\}}.$$

As  $\dim_{\mathbf{Q}} \langle 1, \theta, \xi \rangle = 3$  by assumption, the sequence  $(\{n\theta\}, \{n\alpha\})$  is uniformly distributed in  $[0, 1]^2 \subset \mathbf{R}^2$  (Lemma 5.5). We claim that the function

$$f(s, t) := \log |1 - e^{2\pi i s}| \cdot e^{2\pi i m t}$$

lies in  $\text{BSU}^2(\{0, 1\})$  for the singular weight  $g(s) := \log |1 - e^{2\pi i s}|$  and  $h(s, t) := e^{2\pi i m t}$ . Since  $g$  is the special case  $m = 0$  of the function in Equation 6.9, we have already settled that  $g \in \text{BS}(\{0, 1\})$  and  $h$  is clearly Riemann-integrable on  $[0, 1]^2$ . Moreover, the condition of Equation 6.11 only needs to be verified for the singular weight, which we have also already done in the proof of Claim 1. Thus, Theorem 5.12 applies and we get

$$= \int_0^1 \int_0^1 \log |1 - e^{2\pi i s}| \cdot e^{2\pi i m t} ds dt = \left( \int_0^1 \log |1 - e^{2\pi i s}| ds \right) \cdot \left( \int_0^1 e^{2\pi i m t} dt \right).$$

The first integral, by definition, agrees with  $W_0$ , but we already know that  $W_0 = 0$  by Lemma 6.4 (in particular, we do not even need to use that for  $m \neq 0$ , the second factor also vanishes). This finishes the proof.  $\square$

## 7. LIMIT VALUES NEAR THE UNIT CIRCLE

In this section we shall handle the only remaining case:  $R_x$  for  $|x| = 1$  and  $x$  is not a root of unity.

**Theorem 7.1.** *Suppose  $x \in \mathbf{C}$  is an algebraic integer with  $|x| = 1$  and  $x$  is not a root of unity. Then for every point  $p \in \wp$  in  $\wp := \{x^m \mid m \in \mathbf{Z}\}$ , we have*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|) R_x(z) = -\frac{1}{2|m|} \delta_{m \neq 0}.$$

*In particular, the unit circle is the natural boundary for  $R_x$ . For any fractional exponent  $m$  one still has*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|) R_x(z) \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

where  $\chi$  runs through a finite set (depending on  $p$ ) of non-principal Dirichlet characters of various moduli. Write  $x = e^{2\pi i\theta}$ . Then for every point  $p = e^{2\pi i\alpha}$  such that  $1, \theta, \alpha$  are  $\mathbf{Q}$ -linearly independent inside the real numbers, we have

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)R_x(z) = 0.$$

It will be more natural to handle this type of result by working with the argument  $\theta$  as opposed to  $x = e^{2\pi i\theta}$  itself, so we switch to this viewpoint in this section: For  $\theta \in (0, 1)$  irrational, we define the power series

$$Y_\theta(z) := \sum_{n \geq 1} \log |1 - e^{2\pi i n \theta}| \cdot z^n.$$

We need a special notation: For a point  $p$  on the unit circle, a sequence  $(z_n)$  converges *radially* to  $p$  if  $\arg z_n$  is constant for all sufficiently large  $n$ . The corresponding concept of limit is

$$(7.1) \quad \lim_{z \rightarrow p} f(z_n) := \lim_{\substack{z := r \cdot p \\ r \rightarrow 1, r \in (0, 1)}} f(z).$$

The case of interest for this definition are functions  $f$  which vary wildly with the argument.

Based on Theorem 6.1, we can now prove the following characterization of the radial limit values when we approach the unit circle:

**Theorem 7.2.** *Suppose an irrational  $\theta \in (0, 1)$  is given. Moreover, assume*

- $u := e^{2\pi i\theta}$  is an algebraic integer, or
- $\theta$  is badly approximable.

*For every point  $p \in \wp$  with  $\wp := \{e^{2\pi i m \theta} \mid m \in \mathbf{Z}\}$ , we have*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)Y_\theta(z) = -\frac{1}{2|m|} \delta_{m \neq 0}.$$

*For any fractional exponent  $m$  one still has*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)Y_\theta(z) \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

*where  $\chi$  runs through a finite set (depending on  $p$ ) of non-principal Dirichlet characters of various moduli. For every point  $p = e^{2\pi i\alpha}$  such that  $1, \theta, \alpha$  are  $\mathbf{Q}$ -linearly independent inside the real numbers, we have*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)Y_\theta(z) = 0.$$

*In particular, the unit circle is the natural boundary for  $Y_\theta$ .*

If we only wanted the statement about the natural boundary, we could try to invoke the following result:

**Theorem 7.3** (Carroll, Kemperman). *Suppose  $g : [0, 1] \rightarrow \mathbf{C}$  is a Lebesgue-integrable function. Then one and only one of the following statements is true:*

- (1) *For almost all  $\alpha \in \mathbf{R}$ , the power series*

$$F_\alpha := \sum_{n=0}^{\infty} g(\{n\alpha\})z^n$$

*has the unit circle as a natural boundary.*

- (2) *The function  $g$  agrees almost everywhere with a trigonometric polynomial  $\theta \mapsto \sum_{m \in \mathbf{Z}} c_m e^{2\pi i m \theta}$  of period 1.*

This is [CK65, Theorem 1.1]. We can apply this to  $g(t) := \log|1 - e^{2\pi it}|$ . As will be implicit in our proof of Theorem 7.2 below, we can rule out possibility (2), so that (1) must be true. However, this is far too weak for our purposes. We are interested in the case of  $e^{2\pi i\theta}$  being algebraic, which forms a countable set, or  $\theta$  being badly approximable, which is a set of measure zero<sup>4</sup>. Thus, our Theorem 7.2 makes a statement about a set of measure zero. Since the Carroll–Kempman result works only almost everywhere, it is of no help. This is similar to the issue explained in Remark 5.3 that prevents us from exploiting ergodicity directly.

Our proof is based on the following very classical result:

**Lemma 7.4** (Frobenius, [Mor65, Lemma 1]). *Suppose  $C \in \mathbf{C}$  and  $(a_n)$  is a sequence of complex numbers with*

$$\lim_{N \rightarrow \infty} \frac{1}{N} (a_1 + \cdots + a_N) = C.$$

*Define a power series*

$$F(z) := \sum_{n \geq 1} a_n z^n.$$

*If  $F$  has radius of convergence  $\geq 1$ , then  $\lim_{r \rightarrow 1, r \in (0,1)} (1-r)F(r) = C$ .*

While this describes the behaviour of a radial limit point at  $z = 1$ , the idea is that by ‘rotating’ a given function, we can bring any point on the unit circle to lie at  $z = 1$  and apply Frobenius’ Lemma there.

*Proof.* (1) For any point in  $\wp = \{e^{2\pi im\theta} \mid m \in \mathbf{Z}\}$ , let  $m$  be chosen accordingly. We have

$$(7.2) \quad Y_\theta(z e^{2\pi im\theta}) = \sum_{n \geq 1} \log|1 - e^{2\pi in\theta}| (e^{2\pi im\theta})^n z^n$$

and this is itself a power series in  $z$ , which we may temporarily denote by  $V(z)$ . By Lemma 2.4 the series  $V$  has radius of convergence  $\geq 1$ . Then it follows that

$$(7.3) \quad \lim_{\substack{z := r e^{2\pi im\theta} \\ r \rightarrow 1, r < 1}} (1 - |z|) Y_\theta(z) = \lim_{r \rightarrow 1, r \in (0,1)} (1 - r) Y_\theta(r e^{2\pi im\theta}) = \lim_{r \rightarrow 1, r \in (0,1)} (1 - r) V(r).$$

By Lemma 7.4, this limit equals

$$= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (n\text{-th coefficient of } V) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log|1 - e^{2\pi in\theta}| e^{2\pi imn\theta} = -\frac{1}{2|m|} \delta_{m \neq 0}$$

by the Orthogonality Theorem, Theorem 6.1. This proves the first part of the claim. Moreover,  $-\frac{1}{2|m|} \delta_{m \neq 0}$  is non-zero for all  $m \neq 0$ . In particular,  $\wp \setminus \{1\}$  lies in the set of singular points on the radius of convergence. As  $\wp$  is already dense in the unit circle, the same must be true for the set of singular points. Thus, the unit circle is the natural boundary of the power series. For fractional  $m$ , use the corresponding statement of Theorem 6.1.

(2) Now suppose that a point  $p = e^{2\pi i\alpha}$  is given such that  $1, \theta, \alpha$  are  $\mathbf{Q}$ -linearly independent inside the real numbers. The idea of the following proof is taken from the proof of [KN74, Ch. I, Theorem 6.6]. As in line 7.2, we ‘rotate’ the function  $Y_\theta(z)$ : This time, consider

$$Y_\theta(z e^{2\pi i\alpha}) = \sum_{n \geq 1} \log|1 - e^{2\pi in\theta}| (e^{2\pi i\alpha})^n z^n$$

---

<sup>4</sup>measure zero, but uncountable (the badly approximable numbers can be identified with the set of bounded sequences by using the partial quotients).



and write  $V$  for this function, viewed as a power series in  $z$ . Proceed as in line 7.3 and again by Lemma 7.4, the limit turns out to be

$$= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1 - e^{2\pi i n \theta}| e^{2\pi i m n \alpha}.$$

However, this vanishes by Orthogonality, Theorem 6.1.  $\square$

Clearly this theorem immediately implies Theorem 7.1.

## 8. PROOF OF THE MAIN THEOREMS

As we had explained in the introduction, the tools of the previous sections can be used in quite varied applications. This has to do with the fact that the underlying counting problem has shown up in a variety of contexts, which often have no immediate philosophical connection, yet on a technical level lead to formally entirely equivalent problems.

**8.1. Classical knots.** Let  $K \subset S^3$  be a knot<sup>5</sup> and  $X_K$  the knot exterior, i.e.  $X_K$  is a compact real 3-manifold with boundary. One always has

$$H_1(X_K, \mathbf{Z}) \simeq \mathbf{Z}.$$

This isomorphism is not canonical, there are two possible choices, but a posteriori it turns out that the choice does not matter. The Hurewicz Theorem gives us a canonical surjection

$$\pi_1(X_K, *) \twoheadrightarrow H_1(X_K, \mathbf{Z}) \cong \mathbf{Z},$$

which is just the abelianization of the fundamental group. As quotients of the fundamental group correspond to Galois covering spaces with the corresponding deck transformation action, this surjection defines an infinite covering  $X_\infty \rightarrow X$  with a canonical  $\mathbf{Z}$ -action, as well as finite coverings  $X_r \rightarrow X$  with  $\mathbf{Z}/r\mathbf{Z}$ -actions. The  $\mathbf{Z}$ -action on  $X_\infty$  induces an action to homology, so the finitely generated group  $H_1(X_\infty, \mathbf{Z})$  carries an action by the group ring of  $\mathbf{Z}$ . Hence, it is canonically a  $\mathbf{Z}[t, t^{-1}]$ -module. By classical work of Alexander, this module structure has a rather simple structure, namely

$$H_1(X_\infty, \mathbf{Z}) \cong \mathbf{Z}[t, t^{-1}]/(\Delta_K) \quad \text{with} \quad \Delta_K \in \mathbf{Z}[t, t^{-1}].$$

The element  $\Delta_K$  is the *Alexander polynomial*. It is only well-defined up to a unit  $\mathbf{Z}[t, t^{-1}]^\times \simeq \langle \pm 1 \rangle \times t^{\mathbf{Z}}$ . Various normalizations are possible, but for us any choice of a representative in  $\mathbf{Z}[t]$  will be fine<sup>6</sup>.

**Definition 8.1.** We call the roots of  $\Delta_K$  the Alexander roots. Such a root  $\beta$  is called diophantine if  $|\beta| = 1$  and  $\beta \notin \mu_\infty$ , i.e. if it lies on the unit circle, but is not a root of unity.

We prove a refinement of the Silver–Williams theorem [SW02, Theorem 2.1] in the case of knots. There is a fundamental dichotomy, depending on whether there is a diophantine Alexander root or not. Let us begin with the (typical) case in which there is no diophantine root.

<sup>5</sup>for us, a knot is always a tame knot embedded into the 3-sphere.

<sup>6</sup>This is unnatural from the viewpoint of skein relations, but more convenient ring-theoretically.

**Theorem 8.2.** *Let  $K \subset S^3$  be a knot and  $\Delta_K$  its Alexander polynomial. If each root  $\beta_i$  of  $\Delta_K$  either has absolute value  $|\beta_i| \neq 1$  or is a root of unity, then the generating function of torsion homology growth*

$$E_K(z) := \sum_{r=1}^{\infty} \log |H_1(X_r, \mathbf{Z})_{\text{tor}}| \cdot z^r$$

*has radius of convergence 1. However,*

- (1)  $E_K$  admits a meromorphic continuation to the entire complex plane.
- (2) Its poles are located at all integer powers of roots of  $\Delta_K$  which lie outside the open unit disc, i.e.

$$\{\beta^n \mid \Delta_K(\beta) = 0 \text{ and } n \in \mathbf{Z}\} \setminus (\text{open unit disc}).$$

*At  $z = 1$  the pole has order 1 or 2. All other poles are of order 1.*

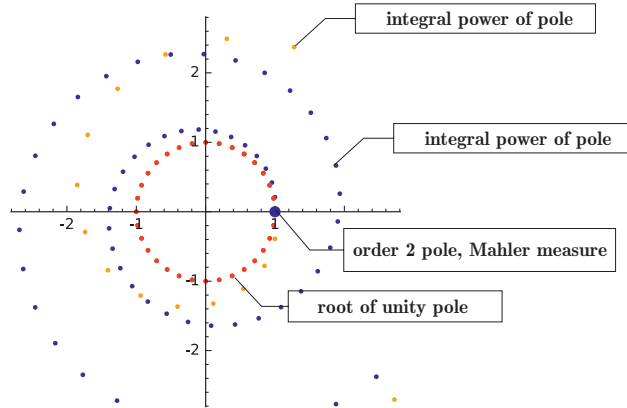
- (3) *The Laurent expansion at  $z = 1$  begins with*

$$\begin{aligned} E_K(z) &= \frac{1}{(z-1)^2} \log \mathcal{M}(\Delta_K) \\ &+ \frac{1}{z-1} \left( \log \mathcal{M}(\Delta_K) + \sum_{\substack{\beta_i \in \mu_{\infty} \\ m := \text{ord}(\beta_i)}} \frac{1}{m} \log \left( \frac{1}{m} \right) \right) \\ &- \sum_{\beta_i, |\beta_i| \neq 1} \log |F(\beta_i^{\pm})| - \sum_{\substack{\beta_i \in \mu_{\infty} \\ m := \text{ord}(\beta_i)}} \left( \frac{m-1}{2} \frac{1}{m} \log \left( \frac{1}{m} \right) + \frac{1}{m} \sum_{l=1}^{m-1} l \cdot \log |1 - \beta_i^l| \right) \\ &+ (z-1) \cdot \text{holomorphic}. \end{aligned}$$

*Here the sums are taken over the roots  $\beta_1, \dots, \beta_n$  of the Alexander polynomial.*

In principle, by assembling our results, we can give the entire Laurent expansion at  $z = 1$  as a closed formula. We leave this to the interested reader.

*Remark 8.3.* It might be worth to sketch the information on the pole loci of the theorem in graphical format. We find



We write “Mahler measure” for the pole at  $z = 1$  as the Theorem shows that we can read off the Mahler measure from the principal part of the Laurent expansion at this point. As we

had explained in §1, the presence of the Mahler measure in this expansion is quite literally equivalent to the Silver–Williams asymptotic.

Under the assumptions of the theorem, we immediately recover the following result:

**Theorem 8.4** (Fried [Fri88]). *Let  $K \subset S^3$  be a knot such that its Alexander polynomial  $\Delta_K$  has no roots in  $\mu_\infty$ . Then knowing the values  $H_1(X_r, \mathbf{Z})_{\text{tor}}$  for all  $r \geq 1$  uniquely determines  $\Delta_K$ .*

We point out that Fried’s result does not come with an easy description how  $\Delta_K$  is to be recovered from the homology torsion cardinalities.

*New proof of special case.* Suppose  $\Delta_K$  has no roots on the unit circle. By our Theorem, the poles of  $E_K$  tell us all integer pole powers, so since  $\mathbf{Z}$  has only two possible generators,  $+1$  or  $-1$ , we can reconstruct  $z$  or  $z^{-1}$  for each root of  $\Delta_K$ . However, the Alexander polynomial is reciprocal, so if  $z$  is a root,  $z^{-1}$  is also a root.  $\square$

Fried’s result has recently found the following application:

**Theorem 8.5** (Boileau–Friedl [BF15, Prop. 4.10]). *Let  $K_1, K_2 \subset S^3$  be knots such that their knot groups have isomorphic profinite completion. If neither Alexander polynomial has a root in  $\mu_\infty$ , both knots have the same Alexander polynomial (up to as unique Alexander polynomials are).*

So, if neither Alexander polynomial has a root on the unit circle, the proof *loc. cit.* also works with Theorem 8.2 instead, but use residual finiteness as in [BF16, Lemma 4.2]. A recent development is due to Ueki [Uek17].

As we had already explained in the narrative of the introduction, everything changes drastically if  $\Delta_K$  *does* have a diophantine root. Then an analytic continuation is impossible. Nonetheless, one can read of a lot of the data which was previously packaged in the poles from the singular values on the radius of convergence:

**Definition 8.6.** *A set of elements  $x_1, \dots, x_r \in \mathbf{C}$ , all on the unit circle, will be called multiplicatively (in)dependent if the real numbers*

$$\{\arg x_1, \dots, \arg x_r\} \subset \mathbf{R}$$

*are linearly (in)dependent over the rationals.*

Moreover, we say that something holds “for all sufficiently divisible  $m$ ” if there exists some integer  $N$  such that the statement holds for all  $n$  which are divisible by  $N$ .

**Theorem 8.7.** *Let  $K \subset S^3$  be a knot and  $\Delta_K$  its Alexander polynomial. If it has at least one diophantine root, then  $E_K$  has the unit circle as its natural boundary. Let  $p$  be a point of the unit circle.*

- (1) *If  $p$  is multiplicatively independent from all diophantine roots, then*

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|) E_K(z) = 0.$$

- (2) *If  $p$  is multiplicatively dependent of the diophantine roots, then for all sufficiently divisible  $m \geq 1$ ,*

$$\lim_{z \rightarrow p^{1/m}} \text{rad}(1 - |z|) E_K(z) \in \mathbf{Q}_{\leq 0}$$

*is a (strictly) negative rational number.*

(3) If  $p$  is multiplicatively dependent of the diophantine roots, then

$$\lim_{z \rightarrow p^m} \text{rad}(1 - |z|) E_K(z) \in \mathbf{Q}(\mu_\infty, \pi, \{L(1, \chi)\}_\chi),$$

where  $\chi$  runs through a finite set (depending on  $p$ ) of non-principal Dirichlet characters of various moduli<sup>7</sup>.

In particular, the rational span

$$\mathbf{Q} \langle (\arg \beta_i)_{\beta_i} \text{ a diophantine root of } \Delta_K \rangle$$

inside the real numbers can be read off the boundary value behaviour of  $E_K$  at the unit circle.

In order to prove Theorem 8.2 and Theorem 8.7, we need some preparations along the lines of [SW02].

*Remark 8.8* (Branched coverings of the 3-sphere). Historically, this story was being looked at from a slightly different perspective. In [Gor72], [SW02] one considers *branched* coverings  $\widehat{X}_\infty$  resp.  $\widehat{X}_r$  over  $S^3$ , instead of the spaces  $X_\infty$  resp.  $X$  over the knot complement. This is explained e.g. [BZ03, Ch. 8, E, §8.18]. They sit in a square

$$\begin{array}{ccc} X_r & \hookrightarrow & \widehat{X}_r \\ \downarrow & & \downarrow \\ X_K & \hookrightarrow & S^3 \end{array}$$

and one has  $H_1(X_r, \mathbf{Z}) \cong \mathbf{Z} \oplus H_1(\widehat{X}_r, \mathbf{Z})$ . See [BZ03, 8.19 (d), Prop.].

**Theorem 8.9** (Fox). *Let  $K \subset S^3$  be a knot. If  $\Delta_K$  has no roots which are roots of unity, the homology groups  $H_1(\widehat{X}_r, \mathbf{Z})$  are finite. In this case,  $|H_1(\widehat{X}_r, \mathbf{Z})| = \prod_{\zeta \in \mu_r} |\Delta_K(\zeta)|$*

This formula is due to Fox [Fox56, §6, (6.1) and (6.3)], modulo a some corrections in the proof due to Weber [Web79].

*Proof of Theorem 8.2 and Theorem 8.7.* Let  $\Delta_K(t) = a \prod_{i=1}^n (t - \beta_i) \in \mathbf{Z}[t]$  be the Alexander polynomial, factored over  $\mathbf{C}$ . According to Remark 8.8 and Fox' formula, Theorem 8.9, we have

(8.1)

$$|H_1(X_r, \mathbf{Z})_{\text{tor}}| = |H_1(\widehat{X}_r, \mathbf{Z})| = \prod_{\zeta \in \mu_r} |\Delta_K(\zeta)| = |a|^r \prod_{i=1}^n \left| \prod_{\zeta \in \mu_r} (\beta_i - \zeta) \right| = |a|^r \prod_{i=1}^n |1 - \beta_i^r|$$

since  $T^r - 1 = \prod_{\zeta \in \mu_r} (T - \zeta)$ . Hence,

$$\begin{aligned} E_K(z) &= \log |a| \cdot \frac{z}{(z-1)^2} + \sum_{i=1}^n \sum_{r=1}^{\infty} \log |1 - \beta_i^r| \cdot z^r \\ &= \log |a| \frac{z}{(z-1)^2} + \sum_{|\beta_i| < 1} R_{\beta_i}(z) + \sum_{|\beta_i| > 1} R_{\beta_i}(z) + \sum_{|\beta_i| = 1} R_{\beta_i}(z). \end{aligned}$$

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<sup>7</sup>Depending on the worst denominator in the multiplicative dependency relation, one can bound the necessary supply of moduli for the required  $\chi$ ; see Theorem 6.1. We leave it to the reader to spell this out.

We use Lemma 3.1 for each summand with  $|\beta_i| > 1$ . This yields

$$(8.2) \quad = \frac{z}{(z-1)^2} \log \mathcal{M}(\Delta_K) + \sum_{|\beta_i| < 1} R_{\beta_i}(z) \\ + \sum_{|\beta_i| > 1} R_{\beta_i^{-1}}(z) + \sum_{\substack{|\beta_i|=1 \\ \beta_i \in \mu_\infty}} R_{\beta_i}(z) + \sum_{\substack{|\beta_i|=1 \\ \beta_i \notin \mu_\infty}} R_{\beta_i}(z),$$

where  $\mathcal{M}(\Delta_K)$  denotes the Mahler measure of the Alexander polynomial. It is independent of the choice of the representative for  $\Delta_K$ . Now, we need a case distinction.

(Case A) Suppose that there is no Alexander root  $\beta_i$  with  $|\beta_i| = 1$  and  $\beta_i \notin \mu_\infty$ . Hence, Equation 8.2 simplifies to

$$(8.3) \quad = \frac{z}{(z-1)^2} \log \mathcal{M}(\Delta_K) + \sum_{|\beta_i| < 1} R_{\beta_i}(z) + \sum_{|\beta_i| > 1} R_{\beta_i^{-1}}(z) + \sum_{\substack{|\beta_i|=1 \\ \beta_i \in \mu_\infty}} R_{\beta_i}(z).$$

By Prop. 2.5 the power series  $R_{\beta_i}$  with  $\beta_i \in \mu_\infty$  admit a meromorphic continuation to the entire complex plane with poles precisely at the finite set  $\{\beta_i^r \mid r \in \mathbf{Z}\}$ , and all these are of order 1. The other summands only feature the power series  $R_\beta$  for a parameter  $\beta$  such that  $|\beta| < 1$ . By Theorem 4.10 any such  $R_\beta$  admits a meromorphic continuation to the entire complex plane whose sole poles are at  $\{\beta^{\mathbf{Z}_{\leq -1}}, \bar{\beta}^{\mathbf{Z}_{\leq -1}}\}$ , each of order one. It follows that the sum of all these analytic continuations is a meromorphic function in all of  $\mathbf{C}$  whose poles are at the following locations: (1) a pole at  $z = 1$  from the initial summand (only if the Mahler measure is  $\neq 1$ ), as well as (2) poles coming from the  $R_{\beta_i}$ , i.e. in total

$$(8.4) \quad \bigcup_{i, |\beta_i| \neq 1} \{\beta_i^{\pm \mathbf{Z}_{\leq -1}}, \bar{\beta}_i^{\pm \mathbf{Z}_{\leq -1}}\} \cup \bigcup_{i, |\beta_i|=1} \{\beta_i^{\mathbf{Z}}\} \cup \{1\},$$

where the first union runs through all roots of the Alexander polynomial which, and we use “+” if  $|\beta_i| < 1$  and “−” if  $|\beta_i| > 1$ . The pole at  $z = 1$  is always in this set because the Alexander polynomial of a knot is always non-trivial, so either the Mahler measure is  $\neq 1$  (so that there is an order 2 pole at  $z = 1$ ), or the Mahler measure is  $= 1$ , but then there must be at least one root at a root of unity and by Prop. 2.5 this also causes a pole at  $z = 1$ . We claim that this set agrees with

$$\{\beta^r \mid \Delta_K(\beta) = 0 \text{ and } r \in \mathbf{Z}\} \setminus (\text{open unit disc}).$$

To see this: In Equation 8.4 all elements lie outside the open unit disc. If we replace all exponents  $\pm \mathbf{Z}_{\leq -1}$  by  $\mathbf{Z}_{\neq 0}$ , then all additional elements we get this way lie inside the open unit disc. Thus, the set in Equation 8.4 agrees with

$$\left( \bigcup_{i, |\beta_i| \neq 1} \{\beta_i^{\mathbf{Z}_{\neq 0}}, \bar{\beta}_i^{\mathbf{Z}_{\neq 0}}\} \cup \bigcup_{i, |\beta_i|=1} \{\beta_i^{\mathbf{Z}}\} \right) \setminus (\text{open unit disc}).$$

All elements in this set are an integral power of an Alexander root since  $\Delta_K$  is a real polynomial, so if  $\beta$  is a solution, so is  $\bar{\beta}$ . The converse inclusion is clear. All the poles coming from the functions  $R_{\beta_i}$  has order 1, so the only possibility to get a pole of higher order is the order 2 pole at  $z = 1$  potentially coming from the initial summand in Equation 8.3 if it is non-zero. This finishes the proof of Theorem 8.2.

(Case B) Suppose there exists at least one Alexander root  $\beta_i$  with  $|\beta_i| = 1$ , but  $\beta_i \notin \mu_\infty$ .

Then Equation 8.2 contains the corresponding summand  $R_{\beta_i}$ . By Theorem 7.1 for all  $p \in \{\beta_i^m \mid m \in \mathbf{Z}\}$ , we have

$$(8.5) \quad \lim_{z \rightarrow p} \text{rad}(1 - |z|)R_{\beta_i}(z) = -\frac{1}{2|m|}\delta_{m \neq 0}.$$

This yields a dense set of singular points of the unit circle, making the unit circle the natural boundary for the summand  $R_{\beta_i}$ . We need to study whether the summation of functions  $R_{\beta_i}$  (for varying  $i$ ) in Equation 8.2 may lead to a cancellation of singular points. We claim that this is not possible, because: (1) Each summand  $R_{\beta_j}$  with  $|\beta_j| < 1$  admits an analytic continuation to the entire complex plane without any poles on the unit circle, so it satisfies

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)R_{\beta_j}(z) = 0.$$

(2) A summand  $R_{\beta_j}$  with  $\beta_j \in \mu_\infty$  only has poles at finitely many roots of unity. Since  $p$  is not a root of unity, we again get  $\lim_{z \rightarrow p} \text{rad}(1 - |z|)R_{\beta_j}(z) = 0$ . (3) Each summand  $R_{\beta_j}$  with  $\beta_j$  multiplicatively independent from  $\beta_i$  (and not a root of unity) also satisfies

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)R_{\beta_j}(z) = 0$$

by the second statement of Theorem 7.1. (4) This only leaves summands  $R_{\beta_j}$  with  $\beta_j$  multiplicatively dependent on  $\beta_i$  as candidates for cancellation. Indeed, by the first part of Theorem 7.1 they may contribute a non-zero value. However, at least after taking a sufficiently divisible power<sup>8</sup>, these may only add up values of the shape

$$-\frac{1}{2|M'|}$$

for suitable  $M' \geq 1$ , i.e. (if  $p$  is a sufficiently divisible power)

$$\lim_{z \rightarrow p} \text{rad}(1 - |z|)E_K(z) = \sum_k \left( -\frac{1}{2|M'_k|} \right) \quad (\text{finite sum}).$$

Along with Equation 8.5, all these values are  $< 0$ , so no non-empty sum of them can be zero. In particular, no cancellation is possible. It follows that  $E_K$  has a dense set of singular points on its radius of convergence. Hence, the unit circle is the natural boundary for this power series. Along the way, we have shown the claimed behaviour at boundary values. This finishes the proof of Theorem 8.7.  $\square$

For the sake of completeness, let us also state a structure result regarding the torsion homology order along with a (rather innocent) bound on the error:

**Theorem 8.10.** *Let  $K \subset S^3$  be a knot and suppose the Alexander polynomial has no diophantine roots. Let  $m$  be the least common multiple of all orders of roots of unity which are roots of  $\Delta_K$ , and  $m = 1$  if there are none. Then there exists an  $m$ -periodic sequence  $(a_r)_{r \geq 0}$ , i.e.*

$$a_{r+m} = a_r \quad \text{for all } r \geq 0$$

such that

$$|\log |H_1(X_r, \mathbf{Z})_{\text{tor}}| - (a_r + \log \mathcal{M}(\Delta_K)r)| \leq \sum_{\beta, |\beta| \neq 1} \frac{|\beta^{\pm 1}|^r}{1 - |\beta^{\pm 1}|^r},$$

where we take  $\beta$  if  $|\beta| < 1$  and  $\beta^{-1}$  if  $|\beta| > 1$ .

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<sup>8</sup>More precisely: We only want integral powers of Alexander roots, so the exponent must be sufficiently divisible to clear all denominators in the multiplicative dependency relation.

*Proof.* We have

$$\begin{aligned} E_K(z) &= \sum_{\beta, |\beta| < 1} R_\beta(z) + \sum_{\beta, |\beta| > 1} R_\beta(z) + \sum_{\beta, |\beta| = 1} R_\beta(z) \\ &= \frac{z}{(z-1)^2} \log \mathcal{M}(\Delta_K) + \sum_{\beta, |\beta| < 1} R_\beta(z) + \sum_{\beta, |\beta| > 1} R_{\beta^{-1}}(z) + \sum_{\beta, |\beta| = 1} R_\beta(z), \end{aligned}$$

where  $\beta$  runs through the roots of the Alexander polynomial. By assumption each root  $\beta$  with  $|\beta| = 1$  is a root of unity, say of  $m$ -th order, and thus (by the definition of  $R_x$ , Definition 2.1) the coefficients in the power series expansion of  $R_\beta$  at  $z = 0$  are periodic of period  $m$ . Thus, taking the least common multiple of these orders, we can split off the summand  $\sum_{\beta, |\beta| = 1} R_\beta(z)$  and encode it as the sequence  $(a_i)_{i \geq 0}$  in our claim. Moreover,

$$\frac{z}{(z-1)^2} \log \mathcal{M}(\Delta_K) = \log \mathcal{M}(\Delta_K) \sum_{r \geq 1} r z^r,$$

so we can also understand the contribution of this summand to the coefficients easily. Next, note that

$$(8.6) \quad |\log |1-x|| \leq \frac{|x|}{1-|x|}.$$

(By the two-sided triangle inequality

$$(8.7) \quad |1-|x|| \leq |1-x| \leq 1+|x|.$$

Note that  $s \mapsto |\log s|$  is monotonously decreasing for real  $s \in (0, 1]$  and monotonously increasing for  $s \geq 1$ . The case  $x = 0$  is trivial, so let us first look at the case  $0 < |x| < 1$ : We need a further case distinction: (Case A)  $|1-x| \in (0, 1]$ . Then Equation 8.7 implies  $|\log(1-|x|)| \geq |\log |1-x||$ . For any real number  $t > -1, t \neq 1$  one has the classical inequality  $\frac{t}{t+1} < \log(1+t)$ , so plugging in  $-|x| \in (-1, 1) \setminus \{0\}$ , we get  $|\log |1-x|| \leq |\log(1-|x|)| \leq \left| \frac{|x|}{1-|x|} \right|$  and line 8.6 is true. (Case B) Now suppose  $|1-x| \geq 1$ . In this case Equation 8.7 implies  $|\log |1-x|| \leq |\log(1+|x|)|$ . For any real number  $t > 0$ , one has the classical inequality  $\log(t) \leq t-1$ , so  $|\log |1-x|| \leq |\log(1+|x|)| \leq |x|$  and again line 8.6 is true.) Hence,

$$|\log |H_1(X_r, \mathbf{Z})_{\text{tor}}| - (a_r + \log \mathcal{M}(\Delta_K)r)| \leq \sum_{\beta, |\beta| \neq 1} \frac{|\beta^{\pm 1}|^r}{1-|\beta^{\pm 1}|^r},$$

where we take  $\beta$  if  $|\beta| < 1$  and  $\beta^{-1}$  if  $|\beta| > 1$ , and we do not sum anymore over the roots with  $|\beta| = 1$ .  $\square$

We can now use Theorem 8.2 to obtain new ways to isolate the family of knots whose torsion homology is periodic:

**Theorem 8.11.** *Let  $K \subset S^3$  be a knot. The following are equivalent:*

- (1) *The values  $|H_1(X_r, \mathbf{Z})_{\text{tor}}|$  are periodic in  $r$ .*
- (2) *All Alexander roots are roots of unity.*
- (3) *The values  $\log |H_1(X_r, \mathbf{Z})_{\text{tor}}|$  satisfy a linear recurrence equation.*
- (4) *The values  $\log |H_1(X_r, \mathbf{Z})_{\text{tor}}|$  are periodic in  $r$ .*
- (5)  *$E_K$  is a rational function.*
- (6)  *$E_K$  has an analytic continuation to a domain containing  $z = 1$  and a pole of order one there.*

- (7)  $E_K$  has an analytic continuation to the entire complex plane with only finitely many poles.

This is a strengthening of Gordon's classical result [Gor72].

*Proof.* (5  $\Leftrightarrow$  2) Given (5), i.e.  $E_K$  is rational, it admits an analytic continuation to the entire complex plane, so all roots of  $\Delta_K$  on the unit circle are roots of unity. As soon as there is a root  $\beta$  of the Alexander polynomial of absolute value  $|\beta| \neq 1$ ,  $E_K$  has infinitely many poles. As there are only finitely many poles, all roots satisfy  $|\beta| = 1$ , so the previous remark covers all roots, i.e. we get (2). The converse is clear. (1  $\Leftrightarrow$  2) [Gor72]. (1  $\Leftrightarrow$  4) obvious, (5  $\Leftrightarrow$  3) Standard algebra. (6  $\Leftrightarrow$  2) The Mahler measure is +1 since all roots lie on the unit circle and in this case the leading coefficient is  $\pm 1$ , too (for the standard normalized Alexander polynomial representative this follows for example from  $\Delta_K(1) = \pm 1$ ). Thus,  $E_K(z) =$

$$\frac{1}{(z-1)^2} \log \mathcal{M}(\Delta_K) + \frac{1}{z-1} \left( \log \mathcal{M}(\Delta_K) + \sum_{\substack{\beta_i \in \mu_\infty \\ m=\text{primitive order of } \beta_i}} \frac{1}{m} \log \left( \frac{1}{m} \right) \right) + (\dots)$$

simplifies to a pole of order one at  $z = 1$ , because the sum over the strictly negative terms  $\frac{1}{m} \log \left( \frac{1}{m} \right)$  is always non-zero. Conversely, if the pole at  $z = 1$  has order one, we must have  $\mathcal{M}(\Delta_K) = 1$ , so if  $a$  denotes the leading coefficient of  $\Delta_K$ , we get

$$1 = |a| \prod_{\beta_i, |\beta_i| \geq 1} |\beta_i| \quad \Rightarrow \quad \mathbf{Z}_{\geq 1} \ni |a| = \frac{1}{\prod_{\beta_i, |\beta_i| \geq 1} |\beta_i|} \leq 1.$$

Hence, we must have  $|a| = 1$  and all roots lie on the unit circle. If any root were not a root of unity, the analytic continuation around  $z = 1$  cannot exist. Thus, we get (2). The converse is clear. (7  $\Leftrightarrow$  2) As used before, if a root  $\beta$  has absolute value  $|\beta| \neq 1$ , the analytic continuation has infinitely many poles, so all roots lie on the unit circle, and by the existence of an analytic continuation, they must be roots of unity. (2) follows. The converse is again clear.  $\square$

**8.2. Higher-dimensional knots and Reidemeister torsion.** Many variations of this theme are possible: For example, higher-dimensional knots in homology spheres, thanks to work of Porti [Por04]. One would proceed as follows, we only sketch the necessary modifications:

Let  $K^n \subset M^{n+2}$  be a PL  $n$ -knot, where  $M^{n+2}$  is a PL  $(n+2)$ -dimensional homology sphere, e.g. the ordinary sphere  $S^{n+2}$  itself. This is sufficient to ensure that the fundamental group of the complement abelianizes to  $\mathbf{Z}$ , and thus one has a similar construction of cyclic branched coverings

$$\widehat{X}_\infty \longrightarrow \widehat{X}_r \longrightarrow M^{n+2}$$

generalizing those of Remark 8.8.

**Theorem 8.12.** *Let  $K^n \subset M^{n+2}$  be a PL  $n$ -knot, where  $M^{n+2}$  is a PL  $(n+2)$ -dimensional homology sphere. If  $\Delta_{K^n, i}$  denotes the  $i$ -th Alexander polynomial, and none of the  $\Delta_{K^n, i}$  has a root in  $\mu_\infty$ , then the generating function of the Reidemeister torsion*

$$J_{K^n}(z) := \sum_{r \geq 1} \log(\tau_r) \cdot z^r$$



with

$$\tau_r := \prod_{i=1}^n \left| H_i(\widehat{X}_r, \mathbf{Z}) \right|^{(-1)^{i+1}}$$

has the following property:

- (1) If no root of any of the Alexander polynomials  $\Delta_{K^n, i}$  has absolute value 1, the function admits a meromorphic continuation to the entire complex plane. Its poles are located at most at all integer powers of all roots of all  $\Delta_{K^n, i}$  which lie outside the open unit disc.
- (2) If some  $\Delta_{K^n, i}$  has a root of absolute value 1 and no other  $\Delta_{K^n, j}$  (with  $j \neq i$ ) has a root at the same value, then  $J_{K^n}$  has the unit circle as its natural boundary. An analytic continuation beyond the unit circle is impossible.

As before, we can also completely describe the Laurent expansion at  $z = 1$ , including an alternating sum of log-Mahler measures now, and can understand the boundary behaviour in case (2). We leave it to the reader to spell out such details.

The key ingredient would be the work of Porti on identifying Reidemeister torsion with higher Alexander polynomials, specifically:

**Theorem 8.13** (Porti [Por04, Theorem 6.1]). *Let  $K^n \subset M^{n+2}$  be a PL  $n$ -knot, where  $M^{n+2}$  is a PL  $(n+2)$ -dimensional homology sphere. If  $\Delta_{K^n, i}$  denotes the  $i$ -th Alexander polynomial, and none of the  $\Delta_{K^n, i}$  has a root in  $\mu_\infty$ , then*

$$\prod_{i=1}^n \left| H_i(\widehat{X}_r, \mathbf{Z}) \right|^{(-1)^{i+1}} = \prod_{i=1}^{n+1} \prod_{\zeta \in \mu_r} |\Delta_{K^n, i}(\zeta)|^{(-1)^{i+1}}.$$

Now, one may use this formula instead of Fox' formula in the proof of Theorem 8.2 and unravel it as in Equation 8.1 to a statement in terms of functions  $R_x$ . Then

$$\begin{aligned} \log \prod_{i=1}^n \left| H_i(\widehat{X}_r, \mathbf{Z}) \right|^{(-1)^{i+1}} &= \sum_{i=1}^n (-1)^{i+1} \log \left| H_i(\widehat{X}_r, \mathbf{Z}) \right| \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} \left( \log |a_i| + \sum_j \log |1 - \alpha_{i,j}^r| \right) \end{aligned}$$

with  $a_i$  the leading coefficients of  $\Delta_{K^n, i}$  and its  $\alpha_{i,j}$  the roots. The viewpoint changes a little here since instead of the generating function of an individual (torsion) homology group, we now get a generating function for Reidemeister torsion

$$J(z) = \sum_{n=1}^{\infty} \log \left| \tau(\widehat{X}_r) \right| \cdot z^n$$

via the identification of the Reidemeister torsion with the Alexander function, based on Milnor and Turaev, [Tur86, Thm. 1.1.1]. We leave the details and further variations of the same theme to the interested reader. For example, Porti's paper [Por04] goes further, generalizing the formulae for branched cyclic coverings of link complements à la Hosokawa–Kinoshita [HK60] and Mayberry's thesis (see [MM82]).

*Remark 8.14.* I do not know to what extent the different Alexander polynomials can have joint roots. If they have, this opens up the possibility that the corresponding terms  $R_x$  in the expansion of  $J_{K^n}$  cancel out if they come from homology groups of different parity. For example, it could happen that two roots lying on the unit circle annihilate each other

so that  $J_{K^n}$  admits an analytic continuation although roots on the unit circle are present. This is the analytic counterpart of the problem that Reidemeister torsion usually does not allow us to control any individual torsion homology group.

**8.3. Application to cyclic resultants.** Suppose  $f \in \mathbf{C}[t]$  is a polynomial. It comes with a sequence of complex numbers  $(r_m)_{m \geq 1}$  defined by

$$r_m := \text{Res}(f, t^m - 1),$$

where “Res” refers to the resultant of two polynomials. The values  $r_m$  are known as the *cyclic resultants*.

**Example 8.15.** *The classical example stems from the work of Pierce and Lehmer. For  $f(t) = t - 2$ , one has  $r_m = 2^m - 1$  (the Mersenne sequence). Inspired by Mersenne’s method to find large prime numbers, Lehmer suggested the following heuristic principle:*

**Heuristic** (Lehmer). *If  $f$  has Mahler measure “very close to 1”, then sequence  $r_m$  should contain “a lot” of prime numbers. See [EEW00].*

One can rephrase the definition of the  $r_m$  in terms of evaluating  $f$  at roots of unity. Thus, it can be rephrased in a format close to the expression in the formula of Fox, Theorem 8.9, and Fried’s Theorem, Theorem 8.4, might suggest that it could be possible to reconstruct  $f$  from the values  $r_m$ . However, this turns out to be false. In general, the values  $r_m$  do not uniquely pin down  $f$ . Hillar shows that generically we should expect  $2^{\deg(f)-1}$  polynomials with the same cyclic resultants [Hil05, Corollary 1.5]. His paper provides a number of examples of distinct polynomials with equal cyclic resultants. *Loc. cit.* also shows that there is a Zariski dense open in the affine space of all *monic* polynomials of any bounded degree for whose polynomials the cyclic resultants uniquely pin down the polynomial. Work of Hillar and Levine discusses criteria ensuring that agreement of finitely many cyclic resultants (depending on the degree of  $f$ ) is sufficient to prove  $f = g$  [HL07]. Hillar [Hil05] also addresses how to solve the problem of reconstructing  $f$  from  $(r_m)$  algorithmically. This is possible since one ‘just’ has to solve a system of multi-variable polynomial equations, namely

$$r_1 = \text{Res}(f, t - 1), \quad r_2 = \text{Res}(f, t^2 - 1), \quad r_3 = \text{Res}(f, t^3 - 1), \dots$$

If one has an upper bound on the possible degree of  $f$ , such a system can be solved algorithmically using Gröbner basis techniques. However, in general it will have several solutions.

The situation is much simpler for reciprocal polynomials:

**Theorem 8.16** (Hillar [Hil05, Corollary 1.12]). *Suppose  $f, g$  are reciprocal polynomials and none of their roots is a root of unity. Then if their cyclic resultants agree, it follows that  $f = g$ .*

This generalizes Fried’s Theorem, Theorem 8.4. Since Alexander polynomials are always reciprocal, this explains why Fried’s reconstruction of the Alexander polynomial is always possible from the torsion homology data, while one cannot reconstruct a general polynomial from the cyclic resultants.

We may, nonetheless, apply our methods to a general  $f$ . To this end, we define:

**Definition 8.17.** *Let  $f \in \mathbf{C}[t]$  be a polynomial. Define*

$$T_f(z) := \sum'_{m \geq 1} \log |\text{Res}(f, t^m - 1)| \cdot z^m,$$

where the notation  $\sum'$  means: We omit the  $m$ -th summand if  $\text{Res}(f, t^m - 1) = 0$  (this happens if and only if  $f$  has an  $m$ -th root of unity as one of its roots).

We obtain a meromorphic continuation:

**Theorem 8.18.** *Suppose  $f \in \mathbf{C}[t]$  is a polynomial with roots  $(\beta_i)$ , none of which is diophantine. Then the function  $T_f$  admits a meromorphic continuation to the entire complex plane with poles of order 1 at*

$$(8.8) \quad \bigcup_{i, |\beta_i| \neq 0, 1} \{\beta_i^{\pm \mathbf{Z}^{\leq -1}}, \overline{\beta_i}^{\pm \mathbf{Z}^{\leq -1}}\} \cup \bigcup_{i, |\beta_i|=1} \{\beta_i^{\mathbf{Z}}\}$$

and perhaps a pole of order 1 or 2 at  $z = 1$  (or no pole there).

*Proof.* This is essentially shown as in the proof of Theorem 8.2: If  $f$  factors as  $a \prod_{i=1}^n (t - \beta_i) \in \mathbf{C}[t]$ , then

$$|\text{Res}(f, t^m - 1)| = |a|^m \prod_{i=1}^n |1 - \beta_i^m|.$$

Thus,  $T_f = \log |a| \cdot \frac{z}{(z-1)^2} + \sum_{i \text{ with } \beta_i \neq 0} R_{\beta_i}(z)$ . Now we may proceed as in the proof of Theorem 8.2, with slight modifications. We arrive at

$$= \frac{z}{(z-1)^2} \log \mathcal{M}(f) + \sum_{0 < |\beta_i| < 1} R_{\beta_i}(z) + \sum_{|\beta_i| > 1} R_{\beta_i^{-1}}(z) + \sum_{\substack{|\beta_i|=1 \\ \beta_i \in \mu_\infty}} R_{\beta_i}(z)$$

and can invoke our results about the meromorphic continuation of the functions  $R_\beta$  for  $|\beta| \in (0, 1)$ . We leave the details to the reader.  $\square$

Of course, there is also an analogue of Theorem 8.7 in the case of diophantine roots. We will not spell this out in detail as it is entirely analogous to the treatment in the case of Alexander polynomials for knots.

Whenever the hypotheses of the above theorem are met, we obtain a new proof of the following result of Hillar from 2002:

**Theorem 8.19** (Hillar [Hil05, Theorem 1.8]). *Let  $f, g \in \mathbf{R}[t]$  be polynomials such that their cyclic resultants are all non-zero. Then the absolute values of the cyclic resultants agree, i.e.*

$$(8.9) \quad |r_m(f)| = |r_m(g)| \quad (\text{for } m \geq 1),$$

if and only if there exist  $u, v \in \mathbf{C}[t]$  with  $u(0) \neq 0$  and integers  $\ell_1, \ell_2 \geq 0$  such that

$$f(t) = \pm t^{\ell_1} v(t) u(t^{-1}) t^{\deg u}$$

$$g(t) = t^{\ell_2} v(t) u(t).$$

We shall now give a new proof of this result under slightly more restrictive hypotheses: We need to assume that no root of  $f$  (regarded over the complex numbers) lies on the unit circle. Hillar's condition that all cyclic resultants are non-zero only rules out that no roots of unity appears as roots, so this is a strictly stronger assumption:

*New proof (under this assumption).* Condition 8.9 means that  $T_f = T_g$ . Thus, by Theorem 8.18 for both  $f, g$  the sets of poles

$$\bigcup_{i \text{ with } \beta_i \neq 0} \{\beta_i^{\pm \mathbf{Z}^{\leq -1}}\} \cup \{1\}$$

agree, and so do the residues at these poles. Note that since no root of unity is a root by assumption, we could discard the union  $\bigcup_{i, |\beta_i|=1} \{\beta_i^{\mathbf{Z}}\}$  in Equation 8.8, and since the polynomials are real, the complex conjugate of each root is a root itself, so we could discard

the elements  $\overline{\beta_i}$  in Equation 8.8 as well, since they are contained in the set of all root powers anyway. Since we can read off the multiplicity of a root (or its inverse) from the residue at the pole in  $T_f = T_g$ , we deduce that

$$f = at^{\ell_1} \prod (t - \beta_i^{S_i}) \quad \text{and} \quad g = bt^{\ell_2} \prod (t - \beta_i^{T_i})$$

for suitable choices of  $S_i, T_i \in \{\pm\}$ ,  $\ell_1, \ell_2 \geq 0$  and  $a, b \in \mathbf{R}$ . Now, define  $v(t) := b \prod (t - \beta_i)$  with the product running over all  $\beta_i$  such that  $T_i \neq S_i$  (opposite parity), and  $u(t) = \prod (t - \beta_i)$  running over all  $\beta_i$  such that  $T_i = S_i$  (same parity). One checks that this choice of  $u, v$  settles the claim. This last part of the proof agrees verbatim with Hillar's proof ([Hil05, end of Proof of Theorem 1.1]). The converse is immediate.  $\square$

*Remark 8.20 (Comparison).* Let us compare this to Hillar's proof. Similar to Fried's approach, he studies the analytic properties of a function formed from the cyclic resultants. In their setup, this generating function is always rational, which at first sight might appear more convenient than  $T_f$ . As for  $T_f$ , the poles of their function depend explicitly on the roots one is interested in, however, the dependency is more complicated. Inverting it requires an algebraic technique to compare factorizations in the semi-group ring  $\mathbf{C}[G]$ , with  $G \subset \mathbf{C}^\times$  the subgroup generated by the non-zero roots  $\beta_i$  ([Hil05, §2]). Such a step is not needed since our function  $T_f$  allows us to read off the roots essentially directly. Hillar's method has the advantage that it also works in the (highly non-generic) case of diophantine roots, where our  $T_f$  fails to admit a meromorphic continuation.

**8.4. Exceptional units.** Let  $K$  be a number field and  $u \in \mathcal{O}_K^\times$  a non-torsion unit. Write  $\mathcal{N}$  for the ideal norm. The power series

$$(8.10) \quad G_u(z) := \sum_{r \geq 1} \log \mathcal{N}(1 - u^r) \cdot z^r$$

always has radius of convergence *precisely* 1, and diverges elsewhere. Besides our interest in torsion homology, the function encodes several invariants which have been studied before in different contexts:

If  $u \in \mathcal{O}_K^\times$  is a unit, it is called *exceptional* if  $1 - u$  is also a unit. More geometrically, an exceptional unit is an  $\mathcal{O}_K$ -integral point of  $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ . This is a classical Diophantine problem, and a number of cases have been worked out in the literature, e.g. [Enn91], [NS98]. We shall later need the following non-trivial fact:

**Proposition 8.21** (Siegel). *A number field  $K$  has only finitely many exceptional units.*

Lang shows in [Lan60] how this reduces to Siegel's theorem on the finiteness of integral points of genus  $\geq 1$  curves. The original result of Siegel is [Sie21, Satz 10]. The result was stated in the above form both by Nagell [Nag64, Thm. 8] as well as Chowla [Cho61]. A textbook version including a proof can be found in [HS00, Thm. D.8.1].

**Definition 8.22** (Silverman [Sil95]). *If  $u \in \mathcal{O}_K^\times$  is a unit, denote by  $E(u)$  the number of values for  $n \geq 1$  such that  $1 - u^n$  is also a unit. Equivalently,  $E(u)$  is the number of vanishing coefficients in the power series  $G_u(z)$ .*

By Siegel's finiteness result, Prop. 8.21,  $E(u)$  is well-defined.

**Definition 8.23** (Stewart [Ste12]). *Stewart defines  $E_0(u)$  as the largest integer such that  $1 - u^n$  is a unit for all  $n$  with  $1 \leq n \leq E_0(u)$ , or zero if no such  $n$  exists. Equivalently, the zero of  $G_u(z)$  at  $z = 0$  has order precisely  $E_0(u) + 1$ .*

*Remark 8.24* (Quantitative aspects). There are also quantitative versions of Siegel's and Silverman's results. Notably, Evertse [Eve84, Thm. 1] implies that there are at most  $3 \cdot 7^n$  exceptional units in  $K$ , where  $n := [K : \mathbf{Q}]$ . A result due to Silverman [Sil95] states that there exists an absolute constant  $C$  such that

$$E(u) \leq C \cdot n^{1 + \frac{7}{10 \log \log n}}.$$

Moreover, Stewart [Ste12, Corollary 1] provides the upper bound

$$E_0(u) \leq C' \cdot \frac{n(\log(n+1))^4}{(\log \log(n+2))^3},$$

for some other absolute constant  $C'$ .

As before, we obtain:

**Theorem 8.25.** *Let  $K$  be a number field and  $u \in \mathcal{O}_K^\times$  a unit. Suppose no embedding  $\sigma : K \hookrightarrow \mathbf{C}$  has  $|\sigma u| = 1$ . Then the function  $G_u$  admits a meromorphic continuation to the entire complex plane, with poles at:*

$$\{\text{all Galois conjugates of } u^n \text{ for } n \in \mathbf{Z}\} \setminus (\text{open unit disc})$$

and locally at  $z = 1$ , we have

$$G_u(z-1) = \log \left( \mathcal{M}(u)^{[K:\mathbf{Q}(u)]} \right) \frac{z}{(z-1)^2} - \sum_{\sigma} \log |F(\sigma u^{\pm 1})| + \mathcal{O}(z-1),$$

where  $\mathcal{M}(u)$  is the Mahler measure of  $u$ ,  $F$  the generating function of the partition function,  $\sigma$  runs through all embeddings  $\sigma : K \hookrightarrow \mathbf{C}$ , and “ $\pm$ ” stands for  $+$  if  $|\sigma u| > 1$  and  $-$  if  $|\sigma u| < 1$ .

We leave the proof to the reader; it is just a variation of what we have done for knots. Note that in the case at hand the underlying polynomial is the minimal polynomial. It need not be reciprocal.

**Theorem 8.26.** *Let  $u, v \in \mathcal{O}_K^\times$  be units such that no  $\sigma : K \hookrightarrow \mathbf{C}$  sends either into the unit circle. Then the following are equivalent:*

- (1) Equality  $G_u(z) = G_v(z)$ ,
- (2) The unit  $v$  is Galois conjugate to  $u$  or  $u^{-1}$ .

The infinity of the poles implies that the function  $G_u$  cannot be rational. We deduce:

**Corollary 8.27.** *Let  $u \in \mathcal{O}_K^\times$  be a unit such that no  $\sigma : K \hookrightarrow \mathbf{C}$  sends it into the unit circle. Then the sequence*

$$a_n := \log \mathcal{N}(1 - u^n)$$

*does not satisfy any linear recurrence equation with constant coefficients.*

## 8.5. Further variations.

**Example 8.28.** *By work of Boden and Friedl, one can also count irreducible metabelian representations of  $\pi_1(X_K)$  to  $\mathrm{SL}_n(\mathbf{C})$  in terms of a formula similar to Fox' Formula, Theorem 8.9, so our methods also apply to these values, ranging over  $n$ . See [BF08], Theorem 1.2 and most explicitly Corollary 1.3. We have not worked out the details.*

9. SPECIAL  $L$ -VALUES

There is a well-known relation between (multi-variable log-)Mahler measures and special  $L$ -values. This was realized, first experimentally, by the surprising computations of Smyth in [Smy81], e.g.

$$\mathcal{M}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi)$$

(where  $\chi$  is a certain Dirichlet character) and later theoretically explained through the Beilinson conjectures by Deninger [Den97]. We will not re-tell this story and refer to [Boy98], [Vil99] for explanations. Inspired by this, it feels noteworthy that there is a *genuinely different* way how special  $L$ -values appear in our computations, related to the function  $R_x$  when  $x$  is a root of unity. We may re-interpret Proposition 2.5 as follows:

**Proposition 9.1.** *Suppose  $x \in \mu_m$  is an  $m$ -th root of unity. Then*

$$R_x \in \mathbf{Q}(\zeta_m)(\{L(1, \chi)\}_{\chi \in M})$$

*for some set  $M$  of Dirichlet characters  $\chi$  modulo  $m$ . That is:  $R_x$  is a rational function over a finitely generated field extension of the rationals, generated by the  $m$ -th roots of unity and a finite number of special  $L$ -values of Dirichlet characters at  $s = 1$ .*

*Proof.* Let  $m \geq 2$  be any integer and  $f : \mathbf{Z}/m \rightarrow \mathbf{C}$  be a function. Using the Fourier theory of the group  $(\mathbf{Z}/m, +)$ , we get

$$(9.1) \quad f(n) = \sum_{l=0}^{m-1} \widehat{f}(l) e^{2\pi i \frac{ln}{m}} \quad \text{for} \quad \widehat{f}(n) = \frac{1}{m} \sum_{l=0}^{m-1} f(l) e^{-2\pi i \frac{ln}{m}}.$$

In particular,  $\widehat{f}(0) = \frac{1}{m} \sum_{l=0}^{m-1} f(l)$ . Now, suppose we have  $\widehat{f}(0) = 0$ . In this case, the Dirichlet series associated to  $f$  has the shape

$$L(s, f) = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

Expanding  $f$  as its Fourier series over  $\mathbf{Z}/m$ , this becomes

$$L(s, f) = \sum_{l=0}^{m-1} \widehat{f}(l) \sum_{n \geq 1} \frac{(e^{2\pi i \frac{l}{m}})^n}{n^s} = \sum_{l=0}^{m-1} \widehat{f}(l) L(s, \chi_l)$$

with  $\chi(n) := e^{2\pi i \frac{l}{m}n}$ . Since  $\widehat{f}(0) = 0$  by assumption, only the summands with  $l \neq 0$  appear in the sum, and for these  $\chi_l$  is a non-principal character. Thus, each  $L(s, \chi_l)$  admits a holomorphic continuation to the entire complex plane and the value at  $s = 1$  is given by the convergent series

$$\begin{aligned} L(1, f) &= \sum_{l=1}^{m-1} \widehat{f}(l) L(1, \chi_l) = \sum_{l=1}^{m-1} \widehat{f}(l) \sum_{n \geq 1} \frac{\chi_l(n)}{n} \\ &= \sum_{l=1}^{m-1} \widehat{f}(l) \sum_{n \geq 1} \frac{(e^{2\pi i \frac{l}{m}})^n}{n} = - \sum_{l=1}^{m-1} \widehat{f}(l) \log(1 - e^{2\pi i \frac{l}{m}}). \end{aligned}$$

Now, if all the Fourier coefficients are real, i.e.  $\widehat{f}(l) = \overline{\widehat{f}(l)}$ , then

$$-\frac{1}{2} L(1, f + \bar{f}) = -\frac{1}{2} L(1, f) - \frac{1}{2} \overline{L(1, f)} = \sum_{l=1}^{m-1} \widehat{f}(l) \log \left| 1 - e^{2\pi i \frac{l}{m}} \right|.$$

Given  $1 \leq a \leq m-1$ , define  $\widehat{f}(l) := \delta_{l=a}$  (which determines  $f$  by Fourier inversion). It follows that  $\log \left| 1 - e^{2\pi i \frac{l}{m}} \right| = -\frac{1}{2}L(1, f + \overline{f})$  for this particular  $f$ , and by Fourier expansion,  $f + \overline{f}$  can itself be expanded in terms of characters. Hence, every  $\log \left| 1 - e^{2\pi i \frac{l}{m}} \right|$  is a finite linear combination of special  $L$ -values of Dirichlet characters at  $s = 1$  with coefficients in the cyclotomic field  $\mathbf{Q}(\zeta_m)$  (by Equation 9.1). Thus, Prop. 2.5 implies that

$$R_x \in \mathbf{Q}(\zeta_m)(\{L(1, \chi)\}_{\chi \in M}),$$

where the set  $M$  encompasses the Dirichlet characters appearing in the Fourier expansion of  $f + \overline{f}$ .  $\square$

*Remark 9.2.* Moreover, the somewhat unwieldy expression  $A_m := \sum_{l=1}^{m-1} l \cdot \log |1 - \zeta_m^l|$  in the expansion of  $R_x$  at  $z = 1$  can be interpreted this way. One gets  $A_m = -\frac{1}{2}L(1, f + \overline{f})$  for  $f$  determined by  $\widehat{f}(l) = l$  for  $l = 0, 1, \dots, m-1$ .

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